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THE MOTION OF A VORTEX FILAMENT WITH AXIAL FLOW

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CONTENTS

Infinitesimal waves on a uniform vortex with axial flow are studied. The equation for the frequency of helical waves is obtained, and solved for the case of long waves which leave the internal structure almost unaltered. A method is developed to obtain results for vortices of non-uniform structure and for displacements which are not necessarily small compared with the core radius. The approach consists of balancing the Kutta–Joukowski lift force, the momentum flux due to the axial motion, and the 'tension' of the vortex lines. A general equation for the motion of a vortex filament is obtained, valid for arbitrary shape and internal structure, and in the presence of an external irrotational velocity field. When the axial flow vanishes, the method is equivalent to using the Biot–Savart law for the self-induced velocity, with a suitable cutoff. The impulse of a vortex filament is discussed and its rate of change is given.

1. The dispersion equation for infinitesimal waves on a line vortex

The recent interest in the trailing vortices produced by large aircraft has stimulated work on the motion of line vortices with axial flow. An account of recent developments is found in Olsen, Goldburg & Rogers (1971).

Our object is to obtain approximate equations of motion for a vortex filament containing axial flow and we attack this problem in §§4 to 9 of the present paper. A useful check on the correctness of our approximate equations is provided by comparison with the solutions for waves of infinitesimal amplitude on a straight vortex of circular cross-section, whose vorticity is uniform. In the absence of axial flow the dispersion equation and its form for long waves were discussed by Kelvin (1880). Recently Krishnamoorthy (1966) extended the analysis to include a uniform axial flow in the core and studied the stability of the core numerically, his interest being in vortex breakdown.

In this section we restate Krishnamoorthy's result and consider what sort of waves are permitted. In § 2 we restrict consideration to long helical waves, with a view to determining their propagation speed.

We use cylindrical polars (r, θ, z). The undisturbed flow is a cylindrical vortex with circumferential velocity Ωr for $r \leqslant a$ and $\Omega a^2/r$ for $r \geqslant a$. The axial velocity is W (constant) for $r < a$ and zero for $r > a$. We consider waves in which the boundary of the deformed vortex is

$$r = a + D\,e^{i(\omega t + m\theta + \gamma z)}, \tag{1.1}$$

where $D \ll a$ and where m is an integer and γ is real. Then (Krishnamoorthy 1966) the dispersion equation is

$$\frac{(g-\gamma W)^2}{4\Omega^2 - g^2}\left[\beta a \frac{J'_{|m|}(\beta a)}{J_{|m|}(\beta a)} + \frac{2\Omega m}{g}\right] = -|\gamma|\, a \frac{K'_{|m|}(|\gamma|\, a)}{K_{|m|}(|\gamma|\, a)}, \tag{1.2}$$

where
$$g = \omega + m\Omega + \gamma W \tag{1.3}$$

and
$$\beta^2 = \gamma^2(4\Omega^2 - g^2)/g^2. \tag{1.4}$$

The symmetry properties of Bessel functions of integral order reveal that the order could equally well have been written as m. We have retained the redundant modulus sign to stress that m need not be positive.

Equation (1.2) is a transcendental equation for g. When $W = 0$, it reduces to the equation given by Kelvin for oscillations of a uniform vortex. For $\Omega = 0$, $W \neq 0$, it can be shown that it reduces to that obtained by Batchelor & Gill (1962) for small disturbances to a uniform circular jet.

A general discussion of the roots of equation (1.2) for arbitrary γ and m is beyond our capabilities. It is clear on physical grounds and can be verified analytically that there are complex roots for large γ if $W \neq 0$, corresponding to the Kelvin–Helmholtz instability of the boundary. There is also the spectrum of inertial waves in the rigidly rotating core. However, when considering the motion of the core as a whole rather than its small-scale internal structure, it is reasonable to believe that only the long waves, $|\gamma|\, a \ll 1$, will matter. And, moreover, for this purpose it is the helical disturbances with $m = \pm 1$ which are of most interest. We shall therefore confine attention, henceforth, to the case $|\gamma|\, a \ll 1$ and $m = \pm 1$.

Equation (1.2) leads to a dispersion equation

$$\omega = \omega(\gamma, m) = -\omega^*(-\gamma, -m), \tag{1.5}$$

where * denotes the complex conjugate. If ω is an even function of γ, two solutions may be superimposed to give a motion in which the boundary of the vortex is

$$r = a + 2D\cos\gamma z\, e^{i(\omega t + m\theta)}, \tag{1.6}$$

so that the centre line of the vortex lies in a rotating plane and the disturbance does not propagate along the vortex. Likewise, if ω is independent of the sign of m, two solutions may be superimposed to give the boundary

$$r = a + 2D\cos\theta\,\mathrm{e}^{\mathrm{i}(\omega t+\gamma z)}, \tag{1.7}$$

so that the centre line of vortex lies in a fixed plane but the wave propagates along the vortex. As we shall see below, the two different conditions on ω are satisfied for the two cases $W = 0$ and $\Omega = 0$, respectively. If $\omega(\gamma, m) \neq \omega(\gamma, -m)$ the two helical solutions can be superimposed to give the boundary

$$r = a + 2D\cos(\theta+\alpha t)\,\mathrm{e}^{\mathrm{i}(\gamma z+\epsilon t)}, \tag{1.8}$$

where
$$\left.\begin{aligned}\epsilon &= \tfrac{1}{2}(\omega(\gamma,1)+\omega(\gamma,-1))\\ \alpha &= \tfrac{1}{2}(\omega(\gamma,1)-\omega(\gamma,-1)).\end{aligned}\right\} \tag{1.9}$$
and

This represents a sinusoidal disturbance of the centre line which propagates and lies in a rotating plane.

Finally if $\omega(\gamma, m) \neq \omega(\gamma, -m)$ and ω is complex no such combination of the two helices is possible because they will, in general, amplify at different rates.

2. Long waves

In the long wave limit, with $m = \pm 1$,

$$-|\gamma|\,a\frac{K_1'(|\gamma|\,a)}{K_1(|\gamma|\,a)} = 1+\gamma^2a^2\ln\frac{2}{a|\gamma|} - C\gamma^2a^2 + O\{\gamma^4a^4(\ln|\gamma|\,a)^2\}, \tag{2.1}$$

where $C = 0.5772\ldots$ is Euler's constant. There are still in this limit an infinite number of modes, for as $g \to 0$ the quantity $\beta a \to \infty$ and passes through the infinity of roots of $J_1(\beta a)$ and it is clear that a solution of the dispersion equation can always be found with βa near one of these roots. However, these modes have highly oscillatory radial dependence, and it is again plausible to suppose that they are of limited physical interest, either because they are hard to excite or are damped by viscosity. The mode of greatest physical interest is that with the least possible radial dependence of the perturbation quantities. In fact this mode proves to have $\beta a \ll 1$, so that the perturbation quantities vary only slightly across the vortex (although the slight variations in the axial vorticity must not be neglected unless only the leading term in the expansion in powers of γa is wanted).† Our procedure is to assume provisionally that the mode of interest has $\beta a \ll 1$, verifying the consistency of this assumption after the solution is found.

Then
$$\beta a\frac{J_1'(\beta a)}{J_1(\beta a)} = 1-\tfrac{1}{4}\beta^2a^2-\tfrac{1}{96}\beta^4a^4+\ldots, \tag{2.2}$$

and the dispersion equation can be written

$$\left(1-\frac{\gamma W}{g}\right)^2\left[\left(\frac{2\Omega m}{g}-1\right)^{-1}-\frac{\gamma^2a^2}{4}-\frac{\gamma^4a^4}{96}\left(\frac{4\Omega^2}{g^2}-1\right)+\ldots\right] = 1+(K-\tfrac{1}{4})\,\gamma^2a^2+\ldots, \tag{2.3}$$

where $m = \pm 1$ and
$$K = \ln\frac{2}{|\gamma|\,a} - C + \tfrac{1}{4}. \tag{2.4}$$

When the terms involving γ^4a^4 are neglected, equation (2.3) is a cubic for g.

† This is the source of some of the discrepancy between the results of Parks (1971) and Widnall, Bliss & Zalay (1971). Parks neglects the variation over the cross-section of the axial vorticity.

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It is now helpful to introduce dimensionless quantities. We write

$$\sigma = \frac{\omega}{\Omega}, \quad S = \frac{\gamma W}{\Omega}, \quad R = \frac{W}{\Omega a} = \frac{S}{\gamma a}. \tag{2.5}$$

Then the dispersion equation can be put into the form, terms smaller than $O(\gamma^2 a^2)$ being neglected,

$$\frac{(\sigma+m)^2}{1-(\sigma+S)^2} = 1 + K\gamma^2 a^2 - \gamma^2 a^2 \frac{S(2\sigma+2m+S)}{4(\sigma+m+S)^2}. \tag{2.6}$$

But only roots which satisfy the consistency condition,

$$a^2(\beta^2+\gamma^2) = \frac{4\gamma^2 a^2 \Omega^2}{g^2} = \frac{4\gamma^2 a^2}{(\sigma+m+S)^2} \ll 1, \tag{2.7}$$

are to be retained.

For trailing vortices, it is expected that R is finite and hence $S = O(\gamma a)$. For this case, the consistent solution of (2.6) is

$$\sigma = \tfrac{1}{2}mK\gamma^2 a^2 - \tfrac{1}{2}mS^2 - \tfrac{1}{2}\gamma^2 a^2 S(K+\tfrac{1}{2}) + \tfrac{1}{2}S^3. \tag{2.8}$$

(The other two roots of the cubic violate the consistency condition.) Note that since S changes sign with γ, sinusoidal disturbances will propagate unless the last two terms are dropped.

The existence of the dispersion equation (2.8) implies an integro-differential equation for the displacement of the centre. This is obtained in appendix C and forms a possible starting-point for a nonlinear theory alternative to that developed below.

On the other hand, if $S = O(1)$ so that $R \gg 1$, and the vortex is a jet with weak swirl, the consistent solutions of (2.6) are

$$\sigma = -\tfrac{1}{2}(m+S) \pm \tfrac{1}{2}\sqrt{(1+2mS-S^2)} + \gamma^2 a^2 \left[\frac{K(\sigma+m)^2 + \frac{1}{2}(\sigma+S)(\sigma+S-m)}{4\sigma+2(m+S)}\right], \tag{2.9}$$

provided that S is not close to the value at which the character of the roots changes, and where in the expression in square brackets we substitute the leading term for σ. It can be shown that a necessary condition for stability of the long waves is

$$|S| < \sqrt{(2)} - 1 + \frac{1}{2\sqrt{2}}\gamma^2 a^2 \left(K + \frac{1+\sqrt{2}}{2}\right). \tag{2.10}$$

In the limit $S \to \infty$, we obtain from (2.9)

$$\frac{\omega}{\gamma W} = -\tfrac{1}{2} \pm \tfrac{1}{2}\mathrm{i} - \tfrac{1}{4}\gamma^2 a^2 (K - \tfrac{1}{2}) - \frac{m}{2S}(1 \pm \mathrm{i}). \tag{2.11}$$

The above derivation of these results requires that the displacement of the core be small compared with the core radius, i.e. $D \ll a$. For vortices of small cross-section, this is an extremely restrictive condition. It is clearly desirable to develop alternative approaches for long wave oscillations of a vortex which violate this condition, so that the displacement of the core is not necessarily small compared with the core radius. We should also like to generalize the results to vortices which do not have uniform distributions of vorticity and axial velocity. One method of procedure for vortex filaments of small core radius is to use the Biot–Savart law for vortex velocity with a suitable cutoff to remove the divergence of the integrals. We shall describe this approach in the next section, but it will be seen that it is only correct to $O(\gamma^2 a^2)$ for small amplitude oscillations. A more general method will be developed in §§ 4 to 7.

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3. An alternative approach using a cutoff

The two modes whose frequencies are given by (2.8) are motions in which the vortex is deformed into a spiral of large pitch, the internal structure of the core being only slightly altered. For if the boundary is given by the real part of (1.9), the core is circular with centre on the line

$$r = D, \quad m\theta + \gamma z + \omega t = 0. \tag{3.1}$$

The points of intersection of this helix with a fixed axial plane advance with velocity $-\omega/\gamma$ in the z-direction, while the point of intersection with a fixed plane normal to the axis rotates with angular velocity $-\omega/m$ in the θ increasing sense. The pitch of the spiral is $(\gamma D)^{-1}$. Hence, an alternative way of finding ω is to calculate the velocity of a spiral of large pitch. It now does not matter if the radius of the cylinder on which the spiral lies is large compared with the radius of the vortex core. That is, we can drop the restriction $D \ll a$; but we shall need $a \ll D + (\gamma^2 D)^{-1}$, the radius of curvature of the helix.

We take $m = 1$ and consider the spiral with vector equation

$$\boldsymbol{r} = \boldsymbol{i}D\cos\theta + \boldsymbol{j}D\sin\theta - \boldsymbol{k}(\theta + \omega t)/\gamma. \tag{3.2}$$

Suppose the strength of the vortex is κ. Then we argue that the velocity of the vortex is given by the Biot–Savart law with a cutoff in the integral to avoid the singularity. Thus the velocity at the typical point $\theta = 0$, when $t = 0$, is

$$\boldsymbol{U} = \frac{\kappa}{4\pi}\int_{[\delta]} \frac{\mathrm{d}\boldsymbol{r} \wedge (\boldsymbol{i}D - \boldsymbol{r})}{|\boldsymbol{i}D - \boldsymbol{r}|^3}, \tag{3.3}$$

where the notation $\int_{[\delta]}$ means that a length $2\delta a$ centred on $\theta = 0$ is omitted from the integral, where a is the core radius. The velocity lies in the cylinder containing the helix, and

$$\omega = -(\boldsymbol{U}\cdot\boldsymbol{j}/D) - \gamma\,\boldsymbol{U}\cdot\boldsymbol{k} \tag{3.4}$$

gives the rotation rate. Equation (3.4) follows from the fact that $\boldsymbol{U} - \partial\boldsymbol{r}/\partial t$ is along the tangent to the helix.

The formula (3.3) is assumed valid whatever the shape of the vortex line, provided its radius of curvature is large compared with the core radius, where the number δ depends only on the distribution of swirl and axial velocity in the vortex core. This is because the singularity which the cutoff removes depends on the local structure. It is now assumed that the value of δ can be found by evaluating the cutoff integral analogous to (3.3) for a vortex ring and comparing with the known result (Saffman 1970). This procedure gives

$$\ln 2\delta = \frac{1}{2} - \frac{4\pi^2}{\kappa^2}\int_0^a v^2 a'\,\mathrm{d}a' + \frac{8\pi^2}{\kappa^2}\int_0^a w^2 a'\,\mathrm{d}a', \tag{3.5}$$

where $v(a')$ and $w(a')$ are the swirl and axial velocities in the core as functions of the distance a' from the core centre. For the uniform vortex,

$$\ln 2\delta = \tfrac{1}{4} + W^2/\Omega^2 a^2. \tag{3.6}$$

We now evaluate the integral (3.3), assuming $\gamma D \ll 1$, and substitute into (3.4). It is found that the leading order terms give

$$\omega = (\kappa\gamma^2/4\pi)\,[K + \tfrac{1}{4} - \ln 2\delta]. \tag{3.7}$$

For $m = -1$, the same expression with the opposite sign is obtained.

For a uniform line vortex, (3.6) and (3.7) give the first two terms of the expression (2.8) for ω/Ω, since $\kappa = 2\pi\Omega a^2$. For general v and w, the result (3.7) can also be obtained from the linearized equations of motion (i.e. with $D \ll a$) by constructing an expansion in powers of γ, but the method is complicated and difficult. (The inclusion of axial velocity to this order into the formula for ω was done first by Widnall *et al.* (1971), who studied the motion of a sinusoidal vortex filament by constructing solutions of the equations of motion in powers of γ.)

It should be noticed that an important part of the argument is the assumption that δ for the spiral is the same as δ for the original straight vortex. This is valid because the change in length is $O(\gamma^2 D^2)$, and it can be demonstrated that the effects of changes in core structure are of second order in the amplitude D. However, for finite amplitude motions of a straight vortex, there is no obvious reason why a and δ should not change, and a consistent application of the 'cutoff' requires further equations for the variations of a and δ. These can only be obtained from a consideration of the internal dynamics. It should be noted further that if the Biot–Savart law with cutoff were used to calculate the infinitesimal oscillations of a steady helical vortex or a vortex ring, the changes in core structure would come in at first order and would need to be calculated for a consistent theory.

Provided the internal structure can be estimated with reasonable accuracy, the 'cutoff' method is relatively simple; but it is still qualitatively unsatisfactory in that it predicts non-propagating sinusoidal oscillations when $S \neq 0$, whereas the more accurate formula (2.8) (for $D \ll a$ of course) shows that they are not possible in the presence of axial velocities. We propose now in the next five sections to develop a more complicated equation for vortex motion, which again is not restricted to $D \ll a$ but predicts ω to the accuracy of (2.8) and can be used for finite amplitude motions. The theory discusses the variations of a and δ as the filament moves.

4. The forces on a vortex filament

Suppose a vortex filament of strength κ has parametric equation

$$\boldsymbol{x} = \boldsymbol{R}(\xi, t), \tag{4.1}$$

where $\boldsymbol{x}$ is the position vector in a fixed coordinate frame, and ξ is a parameter which specifies position along the vortex and will be defined precisely in what follows. We denote distance along the filament by $s = s(\xi, t)$, and the unit tangent vector is denoted by $\boldsymbol{s}$. Then

$$\frac{\partial s}{\partial \xi} = \left|\frac{\partial \boldsymbol{R}}{\partial \xi}\right|, \quad \boldsymbol{s} = \frac{\partial \boldsymbol{R}}{\partial \xi}\bigg/\frac{\partial s}{\partial \xi}. \tag{4.2}$$

We define the velocity of the vortex to be $\partial\boldsymbol{R}/\partial t$. This is arbitrary to the extent of a scalar multiple of $\boldsymbol{s}$, because only the velocity of a moving curve normal to itself is uniquely defined. The vortex moves because of the velocity induced by its own vorticity and any external velocity field which may be present, produced for example by another vortex. The problem is to determine an equation for $\partial\boldsymbol{R}/\partial t$.

Our approach is based on the idea that equations of motion for the filament can be derived by balancing the forces that act on an element of the filament. This idea was suggested to us by Dr S. Widnall (see Widnall & Bliss 1971), but the details of our approach are different. The reason for using the force balance to obtain the equations of motion of the filament is that the calculation of the forces requires less knowledge of the detailed structure of the flow in and near the core of the filament than does a direct attack on the problem through the vorticity equation.

We consider an element Δ of the filament, with length $\mathrm{d}s$ bounded by a curved surface C and plane ends E_1 at ξ and E_2 at $\xi+\mathrm{d}\xi$ which are normal to $\boldsymbol{s}$. Applying the conservation of momentum to Δ gives (recalling the fluid has unit density)

$$\frac{\partial}{\partial t}\int_{\Delta}\boldsymbol{u}\,\mathrm{d}V+\int_{C}p\boldsymbol{l}\,\mathrm{d}A+\int_{E_1+E_2}(p\boldsymbol{l}+\boldsymbol{u}(\boldsymbol{u}^{\mathrm{rel}}\cdot\boldsymbol{l}))\,\mathrm{d}A=0 \tag{4.3}$$

where $\boldsymbol{u}$ is the fluid velocity, $\boldsymbol{u}^{\mathrm{rel}}$ the velocity of the fluid relative to the surface and $\boldsymbol{l}$ denotes the outward normal. The fact that C is material has been used to simplify the integral over the curved portion.

We can write
$$\int_{C}(-p)\,\boldsymbol{l}\,\mathrm{d}A=\mathrm{d}s\,\boldsymbol{F}_{E}, \tag{4.4}$$

then $\boldsymbol{F}_E$ is the force per unit length exerted by the fluid outside the vortex filament on the core boundary. If we denote the $\boldsymbol{s}$ component of $\boldsymbol{u}^{\mathrm{rel}}$ by w we can see that

$$\int_{E_1+E_2}(p\boldsymbol{l}+\boldsymbol{u}(\boldsymbol{u}^{\mathrm{rel}}\cdot\boldsymbol{l}))\,\mathrm{d}A=\mathrm{d}s\frac{\partial}{\partial s}\int_{A(s)}(p\boldsymbol{s}+\boldsymbol{u}w)\,\mathrm{d}A,$$

where $A(s)$ denotes a normal cross-section of the vortex. It is convenient to put

$$\mathrm{d}s\frac{\partial}{\partial s}\int_{A(s)}(p\boldsymbol{s}+\boldsymbol{u}w)\,\mathrm{d}A+\frac{\partial}{\partial t}\int_{\Delta}\boldsymbol{u}\,\mathrm{d}V=-\mathrm{d}s\,\boldsymbol{F}_{I}, \tag{4.5}$$

and think of $\boldsymbol{F}_I$ as being the force per unit length exerted on the core boundary by the fluid inside the vortex filament, the motion being determined by the equation

$$\boldsymbol{F}_E+\boldsymbol{F}_I=0.$$

We shall calculate all the terms from first principles, correct to order $(a/\rho)^2$,† where a is the radius of the core assumed circular to leading order‡ and ρ the radius of curvature of the filament. We shall also suppose that $W=O(\kappa/a)$, where πWa^2 is the axial mass flux along the filament, so that axial and tangential velocities in the core may be comparable.

We now consider the exterior force. The remainder of this section will contain an intuitive calculation of this quantity. The results will be justified in § 5 by using the equations of motion to calculate the pressure field and obtaining the force by integration.

The idea is that, when $a\ll\rho$, the exterior force can be found, as in slender body theory, by regarding the element as part of an infinite cylinder with circulation in a uniform exterior velocity field. This exterior velocity field is thought of roughly as being the residual velocity of the fluid at points where $|\boldsymbol{x}-\boldsymbol{R}|\gg a$ but $|\boldsymbol{x}-\boldsymbol{R}|\ll\rho$, when the swirling motion is removed. Each point ξ of the filament has associated with it such an exterior velocity, which we denote by $\boldsymbol{V}(\xi,t)$. We can write

$$\boldsymbol{V}=\boldsymbol{V}_E(\xi,t)+\boldsymbol{V}_I(\xi,t),$$

† In assessing order of magnitude, we throughout regard logarithmic terms which appear in the development as being $O(1)$.

‡ As a further illustration of the difficulties inherent in treating the motion of vortex filaments, it is to be noted that the core need not be circular to leading order. For instance, if the vorticity in the core is uniform, the core could be elliptical and rotate in the manner described by Kirchhoff (see Lamb 1932, § 159). However, if the core is initially circular, it will remain circular to leading order (see Moore & Saffman (1971) for a calculation of the deformation of a vortex by external shear), and for the sake of simplicity only circular cores will be considered.

where $\boldsymbol{V}_{\mathrm{E}}$ is the contribution from external sources which produce an irrotational velocity field $\boldsymbol{U}_{\mathrm{E}}(\boldsymbol{x}, t)$, and $\boldsymbol{V}_{\mathrm{I}}$ is the contribution from the vortex itself. Clearly, it is appropriate to take

$$\boldsymbol{V}_{\mathrm{E}} = \boldsymbol{U}_{\mathrm{E}}[\boldsymbol{R}(\xi, t), t]. \tag{4.6}$$

The problem now is to specify $\boldsymbol{V}_{\mathrm{I}}$, the contribution to the velocity seen by an element which is produced by the rest of the filament. The natural approach would be to define $\boldsymbol{V}_{\mathrm{I}}$ as the velocity given by the Biot–Savart law evaluated at $\boldsymbol{x} = \boldsymbol{R}$, but this integral diverges. Our procedure is merely to adopt that value which is obtained by subtracting in some way the divergent part of the integral. Thus we define the velocity field $\boldsymbol{U}_{\mathrm{I}}(\boldsymbol{x}, t; \xi)$ by

$$\boldsymbol{U}_{\mathrm{I}}(\boldsymbol{x}; \xi) = \frac{\kappa}{4\pi}\int\left\{\boldsymbol{s}' \wedge \frac{(\boldsymbol{x}-\boldsymbol{R}')}{|\boldsymbol{x}-\boldsymbol{R}'|^3}\,\mathrm{d}s' - \boldsymbol{s}_{\odot} \wedge \frac{(\boldsymbol{x}-\boldsymbol{R}_{\odot})}{|\boldsymbol{x}-\boldsymbol{R}_{\odot}|^3}\,\mathrm{d}s_{\odot}\right\}, \tag{4.7}$$

where $\boldsymbol{R}' = \boldsymbol{R}(\xi')$ (the explicit time dependence is henceforth suppressed) and $\boldsymbol{R}_{\odot}(s_{\odot})$ is the osculating circle to the vortex at the point ξ, and take

$$\boldsymbol{V}_{\mathrm{I}}(\xi) = \boldsymbol{U}_{\mathrm{I}}(\boldsymbol{R}(\xi); \xi). \tag{4.8}$$

This is a well-defined velocity defined for each point of the filament, because the integrand in (4.7) is bounded on the filament.

At this stage, we can give the rule specifying ξ. The parameter is chosen so that ξ is constant for a point moving along the vortex with speed $\boldsymbol{V}\cdot\boldsymbol{s}$, i.e.

$$(\partial \boldsymbol{R}/\partial t)\cdot\boldsymbol{s} = \boldsymbol{V}\cdot\boldsymbol{s} = (\boldsymbol{V}_{\mathrm{E}} + \boldsymbol{V}_{\mathrm{I}})\cdot\boldsymbol{s}. \tag{4.9}$$

It is worth mentioning that the contribution to $\boldsymbol{V}_{\mathrm{I}}$ from the osculating circle is perpendicular to $\boldsymbol{s}$, and hence in calculating $\boldsymbol{V}.\boldsymbol{s}$ the Biot–Savart integral could have been used without subtracting the divergence.

We return to the force. The relative velocity of the element and the surrounding fluid is

$$\boldsymbol{V} - \partial\boldsymbol{R}/\partial t = \boldsymbol{Q}(\xi), \quad \text{say}, \tag{4.10}$$

where by virtue of (4.9), $\boldsymbol{Q}$ is perpendicular to $\boldsymbol{s}$. We now assert that the existence of this relative velocity gives a pressure distribution over the surface of the element that produces a force given by the Kutta lift and the effect of the apparent mass of the filament. Thus the force per unit length is

$$\kappa\boldsymbol{Q}\wedge\boldsymbol{s} + \partial(\pi a^2\boldsymbol{Q})/\partial t - \pi a^2\nabla P. \tag{4.11}$$

The first term in (4.11) is the Kutta lift on a vortex filament, and the second term arises from the apparent mass πa^2 per unit length of the vortex filament which has an approximately circular cross-section when $a/\rho \ll 1$. The last term comes from the pressure gradient in the fluid surrounding the element. The necessity for this term is easily seen by considering the motion of a cylinder through a uniformly accelerating fluid. Its calculation is an intricate matter but, fortunately, we need only know that its order of magnitude is given by the convective derivative of $\boldsymbol{V}$.

Now we anticipate from the considerations of § 3 that

$$V_{\mathrm{I}} = O\left(\frac{\kappa}{\rho}\right), \quad \frac{\partial\boldsymbol{R}}{\partial t} = \frac{\kappa}{4\pi\rho}\boldsymbol{b}\ln\left(\frac{\rho}{a}\right) + O\left(\frac{\kappa}{\rho}\right). \tag{4.12}$$

Here $\boldsymbol{b}$ is the unit binormal to the filament, defined by

$$\frac{\partial\boldsymbol{R}}{\partial s} = \boldsymbol{s}, \quad \frac{\partial\boldsymbol{s}}{\partial s} = \frac{\boldsymbol{n}}{\rho}, \quad \boldsymbol{b} = \boldsymbol{s}\wedge\boldsymbol{n}, \quad \frac{\partial\boldsymbol{b}}{\partial s} = -\tau\boldsymbol{n}, \quad \frac{\partial\boldsymbol{n}}{\partial s} = -\frac{\boldsymbol{s}}{\rho} + \tau\boldsymbol{b}, \tag{4.13}$$

where $\boldsymbol{n}$ is the unit normal towards the centre of curvature and τ is the torsion. It will be supposed that the external velocity is at most $O(\kappa/\rho)$. Then

$$\frac{\partial}{\partial t} \approx \frac{\kappa}{\rho^2} \ln \frac{\rho}{a}, \tag{4.14}$$

and the last two terms in (4.11) are $O(a^2/\rho^2)$ relative to the first and will therefore be dropped.

In addition to the Kutta force, there is the force due to the curvature of the filament. The swirling flow around the core is distorted and the flow velocities are slightly increased on the concave side and slightly decreased on the convex side of the filament. The concomitant changes in the pressure field cause a force on the vortex filament acting towards the centre of curvature. We can express this force in the form $T_0 \boldsymbol{n}/\rho$ and, even though it is an exterior force, think of it as being due to a tension† acting in the vortex core.

There is also a contribution to the exterior force due to the fact that the pressure at the core boundary is reduced by the swirling motion below its value at large distances. This leads to a force when the filament is curved. For a hollow vortex, however, the pressure inside the core is constant and equal to the pressure at the boundary, so that the net dynamical effect of this pressure reduction is zero. However, when the core contains a swirling motion there is a further reduction of pressure below its value at the core boundary. If we consider a curved element, we can see that this reduction leads to a suction force acting on the plane ends of the element and this is equivalent to an increase in the effective tension; this is calculated in § 6.

The tension T_0 is not a well-defined physical quantity; its value depends on the precise way in which $\boldsymbol{V}_\mathrm{I}$ is defined, and we emphasize that our considerations apply only to isolated filaments. Our idea is that the force per unit length,

$$\kappa\left(\boldsymbol{V}_\mathrm{E} + \boldsymbol{V}_\mathrm{I} - \frac{\partial \boldsymbol{R}}{\partial t}\right) \wedge \boldsymbol{s} + \frac{T_0 \boldsymbol{n}}{\rho}, \tag{4.15}$$

is a well-defined physical quantity; and when $\boldsymbol{V}_\mathrm{I}$ is specified by a rule (in our case (4.8) but alternatives are possible), then a well-defined value of T_0 exists as a function of κ, a and ρ.‡

The value of T_0 cannot be found directly without detailed calculation, but it can be found indirectly by comparing with the known speed of propagation of a hollow vortex ring (Hicks 1884) of radius ρ and core radius a, namely

$$\frac{\partial \boldsymbol{R}}{\partial t} = \boldsymbol{b} \frac{\kappa}{4\pi\rho}\left[\ln \frac{8\rho}{a} - \frac{1}{2}\right]. \tag{4.16}$$

For a hollow core, the force per unit length (4.15) must vanish, as the only other force on an element is due to uniform pressure over the curved surface and flat ends which has zero resultant. Also, $\boldsymbol{V}_\mathrm{E} = 0$, and $\boldsymbol{V}_\mathrm{I} = 0$ by definition, and hence we deduce from (4.15) that

$$\frac{\partial \boldsymbol{R}}{\partial t} = \boldsymbol{b} \frac{T_0}{\kappa\rho}. \tag{4.17}$$

† We emphasize that we have not proved that the vortex lines in any flow can be thought of as being in a state of tension.

‡ The idea that vortex filaments have a tension and that their motion can be determined by balancing the tension and the Kutta lift force is common talk amongst physicists studying quantized vortex lines in liquid helium (see Hall 1958). However, the development there was based on rough analogy and the idea was not given any formal basis. The use of a force balance for rectilinear vortices in liquid helium was recently employed by Hall (1970).

Comparing (4.16) and (4.17), we obtain

$$T_0 = \frac{\kappa^2}{4\pi}\left[\ln\frac{8\rho}{a}-\frac{1}{2}\right]. \tag{4.18}$$

Note, incidentally, that if there is a surface tension σ for the core surface, the speed of the ring is $(T_0+\pi a\sigma)/\kappa\rho$, for surface tension acts to increase the effective tension in the vortex by $\pi a\sigma$. There is a contribution $2\pi a\sigma$ from the surface tension acting over the circumference, and an additional contribution $-\pi a\sigma$ from the excess pressure σ/a in the core acting over the area πa^2.

5. The exterior forces on a vortex filament

In this section, we shall derive the results (4.15) and (4.18) from first principles by examining in detail the flow around the vortex. First, we introduce an orthogonal curvilinear coordinate system for the vicinity of the filament. Through any point $\boldsymbol{R}(\xi)$ on the filament, there is a plane containing $\boldsymbol{n}$ and $\boldsymbol{b}$. For any point P of space close to the vortex, there is a unique such plane through P containing a neighbouring point on the vortex with parameter ξ. The first coordinate of P is $s(\xi)$, the arc length along the vortex. Now we refer position in the plane to rectangular axes with unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$, such that $\boldsymbol{n}$ makes angle ψ and $\boldsymbol{b}$ makes angle $\frac{1}{2}\pi+\psi$ with $\boldsymbol{i}$. The position vector of P is expressed as

$$\boldsymbol{x}(\mathrm{P}) = \boldsymbol{R}[\xi(s)]+x\boldsymbol{i}+y\boldsymbol{j}. \tag{5.1}$$

It is easy to show that the coordinate system is orthogonal with metric coefficients

$$h_x = 1, \quad h_y = 1, \quad h_s = 1-(x\cos\psi+y\sin\psi)/\rho, \tag{5.2}$$

provided $\psi = \psi(\xi)$ where

$$\mathrm{d}\psi/\mathrm{d}s = \tau. \tag{5.3}$$

This coordinate system has also been used by Ting (1971).

Having obtained these coordinates, we replace x and y by local polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{5.4}$$

and then

$$h_s = h = 1-(r/\rho)\cos(\theta-\psi). \tag{5.5}$$

We are considering swirling flow with circulation κ outside the core, which has boundary $r = a(s,\theta,t)$, and is a material surface. We solve by developing an expansion in $1/\rho$. The velocity potential $\phi(r,s,\theta,t)$ for the motion outside the core satisfies Laplaces equation

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{1}{h}\frac{\partial h}{\partial r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2h}\frac{\partial h}{\partial\theta}\frac{\partial\phi}{\partial\theta}+\frac{1}{h}\frac{\partial}{\partial s}\left(\frac{1}{h}\frac{\partial\phi}{\partial s}\right) = 0. \tag{5.6}$$

Then we write

$$\phi = \phi_0+\phi_1+\phi_2+\ldots,$$

$$a = a_0+a_1+a_2+\ldots, \tag{5.7}$$

where suffix m implies a quantity $O(\rho^{-m})$ and assume additionally that derivatives along the filament are $O(1/\rho)$ relative to those in the $r-\theta$ plane.

The zero-order flow is clearly

$$\phi_0 = \kappa\theta/2\pi \quad \text{for} \quad r \geqslant a_0(s,t). \tag{5.8}$$

The first-order contribution to the potential satisfies

$$\frac{\partial^2\phi_1}{\partial r^2}+\frac{1}{r}\frac{\partial\phi_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi_1}{\partial\theta^2} = -\frac{\kappa}{2\pi\rho r}\sin(\theta-\psi). \tag{5.9}$$

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The boundary condition to be satisfied by ϕ_1 at $r = a_0$ is derived from the condition that the surface of the vortex filament is material. If $\boldsymbol{\Omega}$ is the angular velocity of the triad $\boldsymbol{s}, \boldsymbol{i}, \boldsymbol{j}$ then the velocity of the fluid relative to the coordinate frame defined by (5.1) and (5.3) is

$$\nabla\phi - \partial\boldsymbol{R}/\partial t - \boldsymbol{\Omega}\wedge(\boldsymbol{x}-\boldsymbol{R}),$$

so that the condition that the surface of the filament is material is that on $r = a(s,\theta,t)$

$$\left(\frac{\partial}{\partial t}+\left[\nabla\phi-\frac{\partial\boldsymbol{R}}{\partial t}-\boldsymbol{\Omega}\wedge(\boldsymbol{x}-\boldsymbol{R})\right]\cdot\nabla\right)(r-a(s,\theta,t)) = 0.$$

Now $\partial a/\partial t$ and Ω are $O(1/\rho^2)$ when the time scale is given by (4.14) so that the condition on ϕ_1 becomes

$$\frac{\partial\phi_1}{\partial r} = \frac{\partial\boldsymbol{R}}{\partial t}\cdot[\boldsymbol{n}\cos(\theta-\psi)+\boldsymbol{b}\sin(\theta-\psi)]+\frac{\kappa}{2\pi a_0^2}\frac{\partial a_1}{\partial\theta} \tag{5.10}$$

to be satisfied on $r = a_0(s,t)$.

We need also a condition on ϕ_1 as $r\to\infty$. This is obtained by matching to the inner limit of the outer flow, which for a point P, with position vector $\boldsymbol{x}(\mathrm{P})$, is given by the Biot–Savart law; namely

$$\begin{aligned}\boldsymbol{u} &= \boldsymbol{U}_{\mathrm{E}}+\frac{\kappa}{4\pi}\int\frac{\boldsymbol{s}\wedge(\boldsymbol{x}-\boldsymbol{R})}{|\boldsymbol{x}-\boldsymbol{R}|^3}\,\mathrm{d}s\\ &= \boldsymbol{U}_{\mathrm{E}}+\boldsymbol{U}_{\mathrm{I}}+\frac{\kappa}{4\pi}\int\frac{\boldsymbol{s}_{\odot}\wedge(\boldsymbol{x}-\boldsymbol{R}_{\odot})}{|\boldsymbol{x}-\boldsymbol{R}_{\odot}|^3}\,\mathrm{d}s_{\odot},\end{aligned} \tag{5.11}$$

where $\boldsymbol{R}_{\odot}$ is the osculating circle at ξ. We expand (5.11) in the limit $r \ll \rho$, to obtain

$$\boldsymbol{u} = \frac{\kappa}{2\pi r^2}(X\boldsymbol{b}-Y\boldsymbol{n})+\boldsymbol{V}_{\mathrm{E}}(\xi)+\boldsymbol{V}_{\mathrm{I}}(\xi)+\frac{\kappa}{4\pi\rho}\left[-\frac{XY\boldsymbol{n}}{r^2}-\frac{Y^2\boldsymbol{b}}{r^2}+\boldsymbol{b}\ln\frac{8\rho}{r}\right]+\kappa O\left(\frac{r}{\rho^2}\right), \tag{5.12}$$

where we have written

$$x\boldsymbol{i}+y\boldsymbol{j} = X\boldsymbol{n}+Y\boldsymbol{b}. \tag{5.13}$$

Hence, we deduce that as $r\to\infty$,

$$\phi_1 \sim (\boldsymbol{V}_{\mathrm{E}}+\boldsymbol{V}_{\mathrm{I}})\cdot(X\boldsymbol{n}+Y\boldsymbol{b})+\frac{\kappa}{4\pi\rho}Y\ln\left(\frac{8\rho}{r}\right)+\int(\boldsymbol{V}_{\mathrm{E}}+\boldsymbol{V}_{\mathrm{I}})\cdot\boldsymbol{s}\,\mathrm{d}s, \tag{5.14}$$

where we have used the fact that $h = 1+O(\rho^{-1})$.

It now follows from (5.9), with the boundary conditions (5.10) and (5.14) that

$$\begin{aligned}\phi_1 = {}&(\boldsymbol{V}_{\mathrm{E}}+\boldsymbol{V}_{\mathrm{I}})\cdot(X\boldsymbol{n}+Y\boldsymbol{b})-\frac{a_0^2}{r^2}\left(\frac{\partial\boldsymbol{R}}{\partial t}-\boldsymbol{V}_{\mathrm{E}}-\boldsymbol{V}_{\mathrm{I}}\right)\cdot(X\boldsymbol{n}+Y\boldsymbol{b})+\frac{\kappa Y}{4\pi\rho}\ln\frac{8\rho}{r}\\ &+\frac{\kappa Y a_0^2}{4\pi\rho r^2}\left(\ln\frac{8\rho}{a_0}-1\right)+\int(\boldsymbol{V}_{\mathrm{E}}+\boldsymbol{V}_{\mathrm{I}})\cdot\boldsymbol{s}\,\mathrm{d}s+\phi_1^{\mathrm{def}},\end{aligned} \tag{5.15}$$

where ϕ_1^{def} arises from the deformation a_1. Since we are interested in the force on the filament, and since without loss of generality we can neglect rigid displacements so that the Fourier decomposition of a_1 has no terms proportional to $\sin\theta$ and $\cos\theta$, we can neglect a_1 and ϕ_1^{def} for our present purpose. Henceforth, we shall drop the suffix 0 in a_0, since there is no cause for ambiguity.

We now proceed to calculate the pressure on the surface. In general this would be intricate because the coordinate system is not inertial. However, this only affects the pressure to $O(\kappa^2/\rho^2)$ when the time scale is given by (4.14), and so Bernoulli's equation gives

$$p+\tfrac{1}{2}(\nabla\phi_0)^2+\nabla\phi_0\cdot\nabla\phi_1-\frac{\partial\boldsymbol{R}}{\partial t}\cdot\nabla\phi_0 = O\left(\frac{\kappa^2}{\rho^2}\right), \tag{5.16}$$

where the last term on the left-hand side arises from the fact that the origin of the coordinate system moves with velocity $\partial \boldsymbol{R}/\partial t$, the effect of the rotation being of higher order. The result of the calculation is

$$p = -\frac{\kappa^2}{8\pi^2 a^2} + \frac{\kappa}{\pi a}\left(\frac{Y}{a}\boldsymbol{Q}\cdot\boldsymbol{n} - \frac{X}{a}\boldsymbol{Q}\cdot\boldsymbol{b}\right) - \frac{\kappa^2 X}{4\pi^2 \rho a^2}\left(\ln\frac{8\rho}{a} - \frac{1}{2}\right) + O\left(\frac{\kappa^2}{\rho^2}\right), \tag{5.17}$$

where $\boldsymbol{Q}$ is given by (4.10).

Integrating this pressure over the curved surface of our element of length $\mathrm{d}s$, we obtain the exterior force per unit length

$$\boldsymbol{F}_{\mathrm{E}} = \kappa \boldsymbol{Q} \wedge \boldsymbol{s} + \frac{\kappa^2 \boldsymbol{n}}{4\pi\rho}\left(\ln\frac{8\rho}{a} - \frac{1}{2}\right) - \frac{\kappa^2 \boldsymbol{s}}{8\pi a^2}\frac{\partial a^2}{\partial s} - \frac{\kappa^2 \boldsymbol{n}}{8\pi\rho}. \tag{5.18}$$

The first two terms in (5.18) are the Kutta lift and tension as given in the previous section. The last term in (5.18) comes from the reduction in pressure due to the swirl about the filament. It is cancelled by an equal and opposite term in the interior force $\boldsymbol{F}_{\mathrm{I}}$, and has no consequences for the dynamics. The third term in (5.18) is a force parallel to the filament which arises if a^2 varies along the filament. However, we shall argue below (see § 8) that $\partial a^2/\partial s$ is negligible and this term is unimportant.

Note, incidentally, that the tension term is not in fact the same as would be produced by a true tension, for we can write

$$\boldsymbol{F}_{\mathrm{E}} = \kappa \boldsymbol{Q} \wedge \boldsymbol{s} + \frac{\partial}{\partial s}(T_0 \boldsymbol{s}) - \frac{\kappa^2}{4\pi\rho}\frac{\partial \rho}{\partial s}\boldsymbol{s} - \frac{\kappa^2 \boldsymbol{n}}{8\pi\rho}, \tag{5.19}$$

which shows the equivalence to an actual tension T_0, *plus* a longitudinal force density

$$-(\kappa^2/4\pi\rho)\,\partial\rho/\partial s.$$

Equation (5.18) completes our analysis of the exterior force, but it is necessary before going further to discuss the error. By the method of derivation, it appears that the error is $O(\kappa^2 a/\rho^2)$. However, we shall assert that the error is actually $O(\kappa^2 a^2/\rho^3)$, with logarithmic terms not being explicitly mentioned. The reason is as follows. The force is being calculated as an expansion in powers of a/ρ. Now the sign of ρ is arbitrary. We could have chosen $\boldsymbol{n}$ in the opposite direction to the centre of curvature without invalidating the analysis in any way; ρ would have become negative but the analysis nowhere depends on ρ being of one sign, except that $\ln\rho$ should always be written $\frac{1}{2}\ln\rho^2$. Thus the results must be invariant under the change ρ into $-\rho$. Note that this changes the signs of $\boldsymbol{n}$ and $\boldsymbol{b}$, but leaves the torsion τ unaltered. By inspection, we see that (5.18) has this property provided the error in T_0 is $\kappa^2 O(a^2/\rho^2)$. Detailed calculation to $O(1/\rho^2)$ supports this conclusion, because the angular dependence of p_2 is such that there is no net force on the vortex core.

We conclude this section by examining further the case of a hollow vortex, for which the only quantity not determined is the axial variation of the core radius. Now in the core of a hollow vortex the pressure must be the same everywhere, so that in view of (5.17)

$$a(s,t) = a(t) + O(1/\rho). \tag{5.20}$$

If surface tension is present the core pressure is increased by an amount σ/a, but (5.20) must again hold.

Alternatively, we can consider the force balance, which for the case when there is surface tension in the core boundary is from (5.18)

$$\kappa \boldsymbol{Q} \wedge \boldsymbol{s} + \frac{\kappa^2 \boldsymbol{n}}{4\pi\rho}\left(\ln\frac{8\rho}{a} - \frac{1}{2}\right) - \frac{\kappa^2 \boldsymbol{s}}{8\pi a^2}\frac{\partial a^2}{\partial s} + \frac{\partial}{\partial s}(\pi a \sigma \boldsymbol{s}) = O\left(\frac{1}{\rho^3}\right). \tag{5.21}$$

The tangential component of this equation gives

$$\left(-\frac{\kappa^2}{4\pi a}+\pi\sigma\right)\frac{\partial a}{\partial s}=O\left(\frac{1}{\rho^3}\right),$$

so that

$$a(s,t) = a(t)+O(1/\rho^2). \tag{5.22}$$

This stronger result has resulted from application of the argument about the invariance under sign changes of ρ, but can be confirmed by detailed calculation of the $O(1/\rho^2)$ terms; $a(t)$ must also satisfy the requirement that the volume of the core is conserved, so that a^2L = constant,† where L is the total length of the core.

The condition of force balance (5.21) also implies

$$\frac{\partial \boldsymbol{R}}{\partial t} = \boldsymbol{V}_{\mathrm{E}}+\boldsymbol{V}_{\mathrm{I}}+\frac{\boldsymbol{b}}{\kappa\rho}(T_0+\pi a\sigma). \tag{5.23}$$

and we remark that according to (5.17) when (5.22) and (5.23) are satisfied

$$p-\frac{\sigma X}{\rho a} = \text{constant}+O\left(\frac{1}{\rho}\right) \quad \text{on} \quad r=a, \tag{5.24}$$

so that since the sum of the principal curvatures is $1/a-X/\rho a+O(1/\rho^2)$, the boundary condition on the pressure is satisfied with error $O(1/\rho^2)$.

The hollow vortex is perhaps relevant to hydronautical applications, such as the helical vortices produced by a propellor where the cores are empty because of cavitation. But in aerodynamical applications, the cores are not empty and may in fact have substantial axial velocities, and to determine the equations of motion for these vortex filaments it is necessary to consider the effects of internal motions, i.e. the internal forces.

6. The interior forces

As we showed in § 4 there are two contributions to $\boldsymbol{F}_{\mathrm{I}}$. First there is the force due to the pressure and momentum flux acting across the ends of the filament.

We start with the contribution from pressure. To leading order, the streamlines inside the core are uniform helices relative to the coordinate system (x, y, s), with tangential velocity $v_0(r)$. Likewise, the pressure is to leading order symmetric about the core centre, and

$$\frac{\partial p}{\partial r}=\frac{v_0^2}{r}. \tag{6.1}$$

Then,

$$-\int_0^a 2\pi r p\,\mathrm{d}r = \pi\int_0^a r^2\frac{\partial p}{\partial r}\mathrm{d}r-\pi a^2 p(a)$$

$$= \pi\int_0^a r[v_0(r)]^2\,\mathrm{d}r+\frac{\kappa^2}{8\pi}, \tag{6.2}$$

on using (5.17). Thus the swirling motion inside the core is equivalent to an additional tension $\frac{1}{2}\pi a^2\overline{v^2}$, where $\overline{v^2}$ is the mean value of v_0^2 inside the core. We can calculate explicitly the $O(1/\rho)$ correction to the pressure, but it turns out to give zero net force, as it must be because of the general argument given at the end of § 5 about invariance under change of sign of ρ, and we shall therefore omit the details.

† The volume of a vapour-filled core in a liquid is not constant, and this condition is replaced by the condition that the pressure in the core is equal to the vapour pressure of the liquid.

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Next we consider the momentum flux or Reynolds stress. The velocity of a fluid particle in the core is

$$w\boldsymbol{s}+\boldsymbol{u}_{\perp}+\partial\boldsymbol{R}/\partial t, \tag{6.3}$$

where $\boldsymbol{u}_{\perp}$ is the velocity in the xy-plane relative to the axes and w is the axial velocity relative to a point with constant ξ. The Reynolds stress contribution is therefore

$$-\int w\left(w\boldsymbol{s}+\boldsymbol{u}_{\perp}+\frac{\partial\boldsymbol{R}}{\partial t}\right)\mathrm{d}A = -\pi a^2\overline{w^2}\boldsymbol{s}-\pi a^2\overline{w}\frac{\partial\boldsymbol{R}}{\partial t}-\int w\boldsymbol{u}_{\perp}\,\mathrm{d}A, \tag{6.4}$$

where $\overline{w^2}$ and $\overline{w}$ are the mean values of w^2 and w across the core, and to the accuracy required may be calculated with the leading order velocity field $w_0(r)$, the error being $O(1/\rho^2)$. The calculation of the last term in (6.4) requires the $O(1/\rho)$ correction to the velocity field. It will be shown in appendix A that

$$\int w\boldsymbol{u}_{\perp}\,\mathrm{d}A = \frac{2\pi\boldsymbol{b}}{\rho}\int_0^a r^2 v_0 w_0\,\mathrm{d}r$$

$$= \lambda\kappa\overline{w}a^2\boldsymbol{b}/\rho, \quad \text{say.} \tag{6.5}$$

For a uniform vortex, i.e. w_0 and v_0/r constant, we find that $\lambda = \frac{1}{4}$. The error in (6.4) and (6.5) is $O(1/\rho^2)$.

The second contribution is the rate of change of momentum of the core fluid in the element $\mathrm{d}\xi$ moving with the velocity given by (6.3) and is

$$-\left(\frac{\partial s}{\partial\xi}\right)^{-1}\frac{\partial}{\partial t}\left\{\left(\pi a^2\overline{w}\boldsymbol{s}+\int\boldsymbol{u}_{\perp}\,\mathrm{d}A+\pi a^2\frac{\partial\boldsymbol{R}}{\partial t}\right)\frac{\partial s}{\partial\xi}\right\} = -\pi\frac{\partial}{\partial t}(a^2\overline{w}\boldsymbol{s})-\pi a^2\overline{w}\boldsymbol{s}\frac{\partial}{\partial t}\left(\ln\frac{\partial s}{\partial\xi}\right)-O\left(\frac{1}{\rho^3}\right). \tag{6.6}$$

Combining (6.2), (6.4), (6.5) and (6.6), we have for the internal force per unit length

$$\boldsymbol{F}_{\mathrm{I}} = \pi\frac{\partial}{\partial s}\left[(\tfrac{1}{2}a^2\overline{v^2}-a^2\overline{w^2})\,\boldsymbol{s}-a^2\overline{w}\frac{\partial\boldsymbol{R}}{\partial t}-\lambda\frac{\kappa\overline{w}a^2}{\pi\rho}\boldsymbol{b}\right]+\frac{\kappa^2}{8\pi}\frac{\boldsymbol{n}}{\rho}$$

$$-\pi a^2\overline{w}\frac{\partial\boldsymbol{s}}{\partial t}-\pi\boldsymbol{s}\frac{\partial}{\partial t}(a^2\overline{w})-\pi a^2\overline{w}\boldsymbol{s}\frac{\partial}{\partial t}\left(\ln\frac{\partial s}{\partial\xi}\right). \tag{6.7}$$

We remark that for a hollow core (and no surface tension)

$$\boldsymbol{F}_{\mathrm{I}} = \kappa^2\boldsymbol{n}/8\pi\rho, \tag{6.8}$$

which just cancels the term in $\boldsymbol{F}_{\mathrm{E}}$ due to the pressure reduction.

7. The velocity of a vortex filament (I)

The equation of motion of the filament is obtained by equating the total force to zero, i.e.

$$\boldsymbol{F}_{\mathrm{E}}+\boldsymbol{F}_{\mathrm{I}} = 0, \tag{7.1}$$

where $\boldsymbol{F}_{\mathrm{E}}$ is given by (5.18) and $\boldsymbol{F}_{\mathrm{I}}$ by (6.7). The component perpendicular to $\boldsymbol{s}$ gives

$$\kappa\left(\boldsymbol{V}_{\mathrm{E}}+\boldsymbol{V}_{\mathrm{I}}-\frac{\partial\boldsymbol{R}}{\partial t}\right)\wedge\boldsymbol{s}+\frac{\boldsymbol{n}}{\rho}[T_0+\tfrac{1}{2}\pi a^2\overline{v^2}-\pi a^2\overline{w}^2+\lambda\kappa\overline{w}a^2\tau]-2\pi a^2\overline{w}\frac{\partial\boldsymbol{s}}{\partial t}-\boldsymbol{b}\kappa\frac{\partial}{\partial s}\left(\frac{\lambda\overline{w}a^2}{\rho}\right) = 0, \tag{7.2}$$

the terms neglected being $O(1/\rho^3)$. We have used here the relation (see appendix B)

$$\frac{\partial}{\partial s}\left(\frac{\partial\boldsymbol{R}}{\partial t}\right) = \frac{\partial\boldsymbol{s}}{\partial t}+\boldsymbol{s}\frac{\partial}{\partial t}\left(\ln\frac{\partial s}{\partial\xi}\right). \tag{7.3}$$

and continuity (8.10) is invoked to show $\partial(a^2\overline{w})/\partial s = O(1/\rho^2)$. Remember that $\boldsymbol{s} = \boldsymbol{s}(\xi, t)$, as given by (4.2).

Two checks on (7.2) are immediate. First consider the steady vortex ring in a fluid otherwise at rest. It follows immediately from (7.2) that

$$\frac{\partial \boldsymbol{R}}{\partial t} = \frac{\boldsymbol{b}}{\kappa\rho}[T_0 + \tfrac{1}{2}\pi a^2\overline{v^2} - \pi a^2\overline{w^2}], \tag{7.4}$$

as found in other ways by Widnall *et al.* (1971) and Saffman (1970).

Next we consider the propagation of a uniform helix of large pitch, of the shape given by equation (3.2). By direct calculation, we find that

$$\boldsymbol{V}_{\mathrm{I}} = \frac{\kappa}{4\pi}\gamma^2 D\left(K - \ln\frac{8\rho}{a} + \frac{1}{4}\right)\boldsymbol{b},$$

$$\frac{\partial \boldsymbol{R}}{\partial t} = \omega D\boldsymbol{b}, \quad \frac{\partial \boldsymbol{s}}{\partial t} = \gamma\omega D\boldsymbol{n}, \quad \rho = (D\gamma^2)^{-1}, \quad \tau = -\gamma.$$

Also, $\lambda = \frac{1}{4}$, $\kappa = 2\pi\Omega a^2$, $\overline{v^2} = \frac{1}{2}\Omega^2 a^2$, $\overline{w^2} = W^2$ for a uniform core structure. Then (7.2) leads to

$$\frac{\omega}{\Omega} = \tfrac{1}{2}a^2\gamma^2 K - \frac{1}{2}\frac{W^2\gamma^2}{\Omega^2} - \frac{1}{2}\frac{a^2\gamma^3 KW}{\Omega} + \frac{1}{2}\frac{W^3\gamma^3}{\Omega^3} - \frac{1}{4}\frac{a^2\gamma^3 W}{\Omega}, \tag{7.5}$$

which is in complete agreement with the result for infinitesimal displacements given by equation (2.8). It is to be emphasized that the present derivation does not require $D \ll a$, unlike that of § 2.

We can compare the velocity as given by (7.2) with that calculated using a cutoff as in § 3. The cutoff approximation states that

$$\left[\frac{\partial \boldsymbol{R}}{\partial t}\right]_{\mathrm{cutoff}} = \boldsymbol{V}_{\mathrm{E}} + \frac{\kappa}{4\pi}\int_{[\delta]}\frac{\mathrm{d}\boldsymbol{s}' \wedge (\boldsymbol{R}-\boldsymbol{R}')}{|\boldsymbol{R}-\boldsymbol{R}'|^3}, \tag{7.6}$$

in the notation defined in § 3. We can write this as

$$\left[\frac{\partial \boldsymbol{R}}{\partial t}\right]_{\mathrm{cutoff}} = \boldsymbol{V}_{\mathrm{E}}(\boldsymbol{R}) + \frac{\kappa}{4\pi}\left[\int_{[\delta]}\frac{\mathrm{d}\boldsymbol{s}' \wedge (\boldsymbol{R}-\boldsymbol{R}')}{|\boldsymbol{R}-\boldsymbol{R}'|^3} - \int_{[\delta]}\frac{\mathrm{d}\boldsymbol{s}'_{\odot} \wedge (\boldsymbol{R}-\boldsymbol{R}'_{\odot})}{|\boldsymbol{R}-\boldsymbol{R}'_{\odot}|^3}\right] + \frac{\kappa}{4\pi}\int_{[\delta]}\frac{\mathrm{d}\boldsymbol{s}'_{\odot} \wedge (\boldsymbol{R}-\boldsymbol{R}'_{\odot})}{|\boldsymbol{R}-\boldsymbol{R}'_{\odot}|^3}. \tag{7.7}$$

The second term is easily shown to be $V_{\mathrm{I}}(R) + O(1/\rho^2)$, while a short calculation shows that

$$\frac{\kappa}{4\pi}\int_{[\delta]}\frac{\mathrm{d}\boldsymbol{s}'_{\odot} \wedge (\boldsymbol{R}-\boldsymbol{R}'_{\odot})}{|\boldsymbol{R}-\boldsymbol{R}'_{\odot}|^3} = \frac{\kappa}{4\pi}\frac{\boldsymbol{b}}{\rho}\left(\ln\frac{8\rho}{a} - \ln 2\delta\right) + O\left(\frac{1}{\rho^3}\right). \tag{7.8}$$

Thus substituting the value of δ given in (3.5) we see that $[\partial\boldsymbol{R}/\partial t]_{\mathrm{cutoff}}$ satisfies the equation

$$\kappa\left(\boldsymbol{V}_{\mathrm{E}} + \boldsymbol{V}_{\mathrm{I}} - \left[\frac{\partial \boldsymbol{R}}{\partial t}\right]_{\mathrm{cutoff}}\right) \wedge \boldsymbol{s} + \frac{\boldsymbol{n}}{\rho}[T_0 + \tfrac{1}{2}\pi a^2\overline{v^2} - \pi a^2\overline{w^2}] = 0. \tag{7.9}$$

Thus the cutoff approximation gives the velocity correct to $O(1/\rho)$, but does not take into account the $O(1/\rho)^2$ terms that arise when axial velocities are present. However, it appears that for many practical applications the cutoff approximation is going to be completely adequate.

To summarize the accomplishments so far, we can say that we have obtained an equation (7.2) which gives the velocity of a vortex filament of arbitrary shape and arbitrary internal structure in an external irrotational velocity field. There seems to be no reason in principle why the instantaneous velocity cannot be calculated, numerically if necessary, to arbitrary accuracy.

However, the equation cannot be used to find how a vortex filament moves in time unless we know how the internal structure changes with time, and this information is not contained in (7.2). To complete the theory, we now take up the topic of the internal structure.

8. The internal structure

From the general argument given in §5 about the invariance under changes in sign of ρ, we conclude that changes in a with s are $O(1/\rho^2)$, i.e. we can write

$$a = a(t) + O(1/\rho^2). \tag{8.1}$$

The dependence on t follows from the conservation of volume (if molecular or turbulent diffusion is negligible), and we conclude that

$$La^2 = \text{constant}, \tag{8.2}$$

where L is the total length of the filament.

Likewise, we conclude that the leading order tangential velocity v_0 is independent of s and is a function only of r and t. Conservation of circulation during motion of the filament implies that

$$v_0 = \frac{\kappa}{2\pi r}\Gamma\left(\frac{r}{a}\right), \quad \Gamma(1) = 1, \tag{8.3}$$

where Γ is determined by the initial structure of the vortex. We note that

$$a^2\overline{v^2} = \frac{\kappa^2}{8\pi^2}\mu \quad \text{where} \quad \mu = 4\int_0^1 \frac{1}{\eta}\Gamma^2(\eta)\,\mathrm{d}\eta. \tag{8.4}$$

The quantity μ is constant throughout the motion, and has the value of one for uniform rotation.

The axial velocity w_0 requires more care. Uniform stretching of the vortex preserves the axial velocity profile. On the other hand, external pressure gradients give axial accelerations uniform through the core. In addition, the average axial velocity will vary with position to order $1/\rho$ because of variations of $V_\parallel = \boldsymbol{s}\cdot\boldsymbol{V} = \boldsymbol{s}\cdot\partial\boldsymbol{R}/\partial t$. Thus we have

$$w_0 = W(t) + q(\xi, t) + \frac{\kappa}{b}\chi\left(\frac{r}{a}\right), \tag{8.5}$$

where we can suppose without loss of generality that

$$\int_0^1 \eta\chi(\eta)\,\mathrm{d}\eta = 0. \tag{8.6}$$

Here, q is $O(1/\rho)$, and b is a (constant) length determined by initial conditions. Then

$$\overline{w} = W + q, \tag{8.7}$$

$$a^2\overline{w^2} = a^2W^2 + \frac{a^2\kappa^2}{b^2}\nu + 2a^2Wq + O\left(\frac{1}{\rho^2}\right), \tag{8.8}$$

where

$$\nu = 2\int_0^1 \eta\chi^2(\eta)\,\mathrm{d}\eta \tag{8.9}$$

is a constant determined by the initial conditions.

The velocity q is determined by the equation of continuity for motion in the core. This is

$$\frac{\partial\xi}{\partial s}\frac{\partial'}{\partial t}\left(a^2\frac{\partial s}{\partial \xi}\right) + \frac{\partial}{\partial s}(\overline{w}a^2) = 0, \tag{8.10}$$

i.e.
$$\frac{\partial a^2}{\partial t}+a^2\frac{\partial}{\partial t}\left(\ln\frac{\partial s}{\partial \xi}\right)+a^2\frac{\partial q}{\partial s}=0, \tag{8.11}$$

since $a = a(t)$, $W = W(t)$. Hence, we deduce

$$\frac{\partial q}{\partial s}=-\frac{\partial}{\partial t}\left(\ln\frac{\partial s}{\partial \xi}\right)+\frac{1}{L}\frac{dL}{dt}=-\frac{\partial V_{\parallel}}{\partial s}+\frac{\partial \boldsymbol{R}}{\partial t}\cdot\frac{\boldsymbol{n}}{\rho}+\frac{1}{L}\frac{dL}{dt}, \tag{8.12}$$

where we have used (8.2) and where

$$L=\oint ds,\quad \frac{dL}{dt}=\oint\frac{\partial}{\partial t}\left(\ln\frac{\partial s}{\partial \xi}\right)ds=-\oint\frac{\partial \boldsymbol{R}}{\partial t}\cdot\frac{\boldsymbol{n}}{\rho}ds \tag{8.13}$$

(see appendix B). Note that q is single-valued, i.e.

$$\oint\frac{\partial q}{\partial s}ds=0. \tag{8.14}$$

The main component W of the axial flow is determined by the $\boldsymbol{s}$-component of the force balance equation (7.1). By virtue of (8.1), $\partial a/\partial s = O(1/\rho^3)$, and hence there is no contribution from the external force $\boldsymbol{F}_{\mathrm{E}}$. The internal force $\boldsymbol{F}_{\mathrm{I}}$, given by (6.7), gives a contribution†

$$\pi\frac{\partial}{\partial s}[\tfrac{1}{2}a^2\overline{v^2}-a^2\overline{w^2}]-2\pi a^2\overline{w}\frac{\partial}{\partial t}\left(\ln\frac{\partial s}{\partial \xi}\right)-\pi V_{\parallel}\frac{\partial}{\partial s}(a^2\overline{w})-\pi\frac{\partial}{\partial t}(a^2\overline{w})=O\left(\frac{1}{\rho^3}\right). \tag{8.15}$$

In view of (8.4) and (8.8) this reduces to

$$-2\pi a^2W\frac{\partial q}{\partial s}-2\pi a^2W\frac{\partial}{\partial t}\left(\ln\frac{\partial s}{\partial \xi}\right)-\pi\frac{d}{dt}(a^2W)=0,$$

and substituting for $\partial q/\partial s$ from (8.12)

$$\frac{2a^2W}{L}\frac{dL}{dt}+\frac{d}{dt}(a^2W)=0. \tag{8.16}$$

Hence,
$$Wa^2L^2=\text{constant},\quad\text{or}\quad WL=\text{constant}, \tag{8.17}$$

which can be interpreted as either conservation of angular impulse (see § 10) or conservation of circulation.

The remaining parameter of internal structure is λ (eqn (6.5)). We find that, neglecting q because it is $O(1/\rho)$,

$$\lambda=\int_0^1\left(\eta\Gamma(\eta)+\eta\frac{\kappa}{Wb}\chi(\eta)\,\Gamma(\eta)\right)d\eta+O\left(\frac{1}{\rho}\right). \tag{8.18}$$

Thus λ is constant along the filament, but varies with time because of the dependence on W. If the core is uniform, $\lambda=\frac{1}{4}$.

We conclude that the pertinent properties of the internal structure are determined by the initial conditions and L. The length L is itself determined by the equation of motion which describes how the vortex moves, so the equations are closed.

One difficult and obscure problem remains. It is well known that vortices with axial flow may be subject to vortex breakdown, i.e. sudden changes in the internal structure, and this has been

† The argument of §5 about invariance under $\rho\to-\rho$ does not preclude terms proportional to s/ρ^2 in the external or internal force. However, it is almost obvious that such a term cannot be present in the absence of axial variations and must therefore be proportional to $\partial a/\partial s$ or $\partial\rho/\partial s$. In the former case, it would be negligible because $\partial a/\partial s$ is $O(1/\rho^3)$, and in the latter case it would be like $\kappa^2(a/\rho^2)\,\partial\rho/\partial s$ and violate the invariance under $\rho\to-\rho$.

observed in trailing vortices (Olsen 1971). The relationship of vortex breakdown to the present theory is not understood by the authors. If vortex breakdown occurs it is certainly not true for instance that $a = a(t)$, although a and the other internal parameters W, λ, μ, ν may be piecewise constant. But the equations of filament motion derived in the previous work will not by themselves predict discontinuities in the internal parameters. It seems therefore that vortex breakdown will be associated with a local failure of some or all of the approximations made in this work. One speculation is that the expansions (5.7) fail to converge when vortex breakdown occurs. The physical reason why variations of a are $O(1/\rho^2)$ is that internal waves in the core propagate relative to the vortex with the fast speed κ/a and rapidly equalize variations of a. When $W \approx \kappa/a$, the waves may become trapped and cause a local failure of convergence, but we do not know how to make this idea quantitative in order to predict where breakdown occurs and to determine the changes in internal structure that are produced. The results of this paper will therefore fail to be useful if and when vortex breakdown occurs.

9. The velocity of a vortex filament (II)

We now use the results of § 8 to simplify the equation of motion of the vortex (7.2). We note that

$$a^2\overline{w^2}\frac{\boldsymbol{n}}{\rho}+2a^2\overline{w}\frac{\partial \boldsymbol{s}}{\partial t} = a^2W^2\left(1+\frac{\kappa^2\nu}{W^2b^2}\right)\frac{\boldsymbol{n}}{\rho}+2a^2W\dot{\boldsymbol{s}}+O\left(\frac{1}{\rho^3}\right), \tag{9.1}$$

where we define the quantity $\dot{\boldsymbol{s}}$ by

$$\begin{aligned}\dot{\boldsymbol{s}} &= \frac{\partial \boldsymbol{s}}{\partial t}+q\frac{\boldsymbol{n}}{\rho} = \frac{\partial}{\partial s}\left(\frac{\partial \boldsymbol{R}}{\partial t}\right)-\boldsymbol{s}\frac{\partial}{\partial t}\left(\ln\frac{\partial s}{\partial \xi}\right)+\frac{\partial}{\partial s}(q\boldsymbol{s})-\boldsymbol{s}\frac{\partial q}{\partial s}\\ &= \frac{\partial}{\partial s}\left(\frac{\partial \boldsymbol{R}}{\partial t}+q\boldsymbol{s}\right)-\frac{\boldsymbol{s}}{L}\frac{\mathrm{d}L}{\mathrm{d}t}\end{aligned} \tag{9.2}$$

on using (8.12). The quantity $\dot{\boldsymbol{s}}$ is also given by (appendix B)

$$\dot{\boldsymbol{s}} = \frac{\partial}{\partial s}\left[\frac{\partial \boldsymbol{R}}{\partial t}\right]_s+\frac{s}{L}\frac{\mathrm{d}L}{\mathrm{d}t}\frac{\boldsymbol{n}}{\rho} = \left[\frac{\partial \boldsymbol{s}}{\partial t}\right]_s+\frac{s}{L}\frac{\mathrm{d}L}{\mathrm{d}t}\frac{\boldsymbol{n}}{\rho}, \tag{9.3}$$

where $[\partial \boldsymbol{R}/\partial t]_s$ is the speed of a point moving with constant s.

Hence we write (7.2) as

$$\kappa\left(\boldsymbol{V}_{\mathrm{E}}+\boldsymbol{V}_{\mathrm{I}}-\frac{\partial \boldsymbol{R}}{\partial t}\right)\wedge \boldsymbol{s}+\frac{\boldsymbol{n}}{\rho}\left[T_0+\frac{\kappa^2\mu}{16\pi}-\pi a^2W^2\left(1+\frac{\kappa^2\nu}{W^2b^2}\right)\right]-2\pi a^2W\dot{\boldsymbol{s}}-\lambda\kappa Wa^2\frac{\partial}{\partial s}\left(\frac{\boldsymbol{b}}{\rho}\right) = 0. \tag{9.4}$$

Inversion of this formula gives

$$\frac{\partial \boldsymbol{R}}{\partial t} = \boldsymbol{V}_{\mathrm{E}}+\boldsymbol{V}_{\mathrm{I}}+\boldsymbol{b}\frac{T}{\kappa\rho}-\frac{2\pi a^2W}{\kappa}\boldsymbol{s}\wedge\dot{\boldsymbol{s}}-\lambda Wa^2\boldsymbol{s}\wedge\frac{\partial}{\partial s}\left(\frac{\boldsymbol{b}}{\rho}\right). \tag{9.5}$$

where

$$T = \frac{\kappa^2}{4\pi}\left[\ln\frac{8\rho}{a}-\frac{1}{2}+\frac{\mu}{4}-4\pi^2\left(\frac{a^2\nu}{b^2}+\frac{a^2W^2}{\kappa^2}\right)\right]. \tag{9.6}$$

Now $\dot{\boldsymbol{s}}$ is given by (9.2), where from (8.12)

$$q = -\int^s\left(\frac{\partial V_{\parallel}}{\partial s}-\frac{\partial \boldsymbol{R}}{\partial t}\cdot\frac{\boldsymbol{n}}{\rho}\right)\mathrm{d}s+\frac{s}{L}\frac{\mathrm{d}L}{\mathrm{d}t}, \tag{9.7}$$

but since the term involving $\dot{\boldsymbol{s}}$ is $O(1/\rho^2)$ it is a consistent approximation to evaluate $\dot{\boldsymbol{s}}$ using the leading order expression for $\partial \boldsymbol{R}/\partial t$, i.e. the first three terms on the right-hand side of (9.5). We then find

$$q = -\int^s \boldsymbol{s}\cdot\frac{\partial}{\partial s}(\boldsymbol{V}_\mathrm{E}+\boldsymbol{V}_\mathrm{I})\,\mathrm{d}s + \frac{s}{L}\frac{\mathrm{d}L}{\mathrm{d}t}, \quad \frac{\mathrm{d}L}{\mathrm{d}t} = -\oint (\boldsymbol{V}_\mathrm{E}+\boldsymbol{V}_\mathrm{I})\cdot\frac{\boldsymbol{n}}{\rho}\,\mathrm{d}s, \tag{9.8}$$

and
$$\dot{\boldsymbol{s}} = \frac{\partial}{\partial s}\left(\boldsymbol{V}_\mathrm{E}+\boldsymbol{V}_\mathrm{I}+\boldsymbol{b}\frac{T}{\kappa\rho}\right) - \boldsymbol{s}\left(\boldsymbol{s}\cdot\frac{\partial}{\partial s}(\boldsymbol{V}_\mathrm{E}+\boldsymbol{V}_\mathrm{I})\right) + \frac{\boldsymbol{n}}{\rho}\left[\frac{s}{L}\frac{\mathrm{d}L}{\mathrm{d}t} - \int^s \boldsymbol{s}\cdot\frac{\partial}{\partial s}(\boldsymbol{V}_\mathrm{E}+\boldsymbol{V}_\mathrm{I})\,\mathrm{d}s\right]. \tag{9.9}$$

Thus $\dot{\boldsymbol{s}}(\xi, t)$ is known in terms of $\boldsymbol{R}(\xi, t)$ and the problem of solving (9.5) is one of 'marching' type.

We now have completed the analysis that started in § 4, and we have equations to give the motion of a vortex filament of arbitrary shape and arbitrary internal structure in an external irrotational velocity field, provided the external velocities are small compared with the rotational velocities in the core. The equations are complicated, and exact solutions in closed form will presumably be few. The vortex ring and helix are the only two known at present. However, there is no obvious reason why numerical calculation should not be feasible. If $W = 0$, the velocity given by (9.5) agrees with that obtained by the cutoff approximation. The application of these equations to the effect of axial velocity on the stability of a pair of trailing vortices (Crow 1970) is given in appendix D. Also, the oscillations of a vortex ring are currently under investigation.

10. The linear and angular impulse of a vortex filament

We consider a single closed vortex filament moving under its own induced velocity field in a fluid at rest at infinity. Under these circumstances the net linear impulse $\boldsymbol{I}$ and angular impulse $\boldsymbol{A}$ are constant. In this section we calculate $\boldsymbol{I}$ and $\boldsymbol{A}$ and show that according to our approximate equation of motion (9.5) they are indeed constants of the motion.

The linear impulse $\boldsymbol{I}$ can be calculated as a volume integral over the vorticity distribution $\boldsymbol{\omega}$ from the formula

$$\boldsymbol{I} = \frac{1}{2}\int \boldsymbol{r}\wedge\boldsymbol{\omega}\,\mathrm{d}V. \tag{10.1}$$

If we apply this formula to the closed vortex filament we find, retaining only the two largest orders of magnitude,

$$\boldsymbol{I} = \frac{1}{2}\oint \kappa\boldsymbol{R}\wedge\boldsymbol{s}\,\mathrm{d}s + \pi\oint Wa^2\boldsymbol{s}\,\mathrm{d}s + O(1)$$
$$= \tfrac{1}{2}\kappa\oint \boldsymbol{R}\wedge\boldsymbol{s}\,\mathrm{d}s, \tag{10.2}$$

since Wa^2 = constant along the filament. We have used here the fact that $\mathrm{d}V = \mathrm{d}A\,\mathrm{d}s$, where A is the cross-section area of the filament, and

$$\int \boldsymbol{\omega}\,\mathrm{d}A = \kappa\boldsymbol{s}. \tag{10.3}$$

The second term in (10.2) arises from the vorticity perpendicular to $\boldsymbol{s}$ due to the axial velocity W, the contribution of q being neglected, because it is of higher order.

The rate of change of $\boldsymbol{I}$ can be calculated directly from (10.2), noting that

$$\frac{\partial}{\partial t}(\boldsymbol{s}\,\mathrm{d}s) = \frac{\partial}{\partial s}\left(\frac{\partial \boldsymbol{R}}{\partial t}\right)\mathrm{d}s,$$

and carrying out an integration by parts. The result is

$$\frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}t} = \kappa \oint \frac{\partial \boldsymbol{R}}{\partial t} \wedge \boldsymbol{s}\,\mathrm{d}s. \tag{10.4}$$

This integral can be evaluated on substituting (9.4). We note that

$$\oint \boldsymbol{s}\,\mathrm{d}s = 0, \quad \oint \frac{\boldsymbol{n}}{\rho}\,\mathrm{d}s = 0, \quad \oint \frac{\partial}{\partial s}\left(\frac{\boldsymbol{b}}{\rho}\right)\mathrm{d}s = 0,$$

the first relation coming from (9.2). Further, using the method which leads to (7.9) we can show that

$$\boldsymbol{V}_{\mathrm{I}} + \boldsymbol{b}\frac{T_0}{\kappa\rho} = \frac{\kappa}{4\pi}\int_{[\delta_0]} \frac{\mathrm{d}\boldsymbol{s}' \wedge (\boldsymbol{R}-\boldsymbol{R}')}{|\boldsymbol{R}-\boldsymbol{R}'|^3} + O\left(\frac{1}{\rho^2}\right),$$

with $\delta_0 = \frac{1}{2}\mathrm{e}^{\frac{1}{2}}$. Hence, remembering that $\boldsymbol{V}_{\mathrm{E}} = 0$, (9.5) shows

$$\frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}t} = -\frac{\kappa^2}{4\pi}\oint \mathrm{d}\boldsymbol{s} \wedge \int_{[\delta_0]} \frac{\mathrm{d}\boldsymbol{s}' \wedge (\boldsymbol{R}-\boldsymbol{R}')}{|\boldsymbol{R}-\boldsymbol{R}'|^3}.$$

This is a double integral over the square of side L in the ss'-plane, with a strip along $s = s'$ deleted. The integrand is

$$(\boldsymbol{\Delta}\cdot\mathrm{d}\boldsymbol{s})\,\mathrm{d}\boldsymbol{s}' - \boldsymbol{\Delta}(\mathrm{d}\boldsymbol{s}\cdot\mathrm{d}\boldsymbol{s}'), \quad \boldsymbol{\Delta} = \frac{\boldsymbol{R}-\boldsymbol{R}'}{|\boldsymbol{R}-\boldsymbol{R}'|^3} = -\boldsymbol{\Delta}'.$$

Now $\boldsymbol{\Delta}\cdot\mathrm{d}\boldsymbol{s} = (-\partial|R-R'|^{-1}/\partial s)\,\mathrm{d}s$ and hence the first part of the integrand integrates to a term $O(\rho^{-2})$. The second part gives zero because it is antisymmetric in s and s'. Hence,

$$\mathrm{d}\boldsymbol{I}/\mathrm{d}t = 0. \tag{10.5}$$

When the filament is in an external irrotational velocity field $\boldsymbol{U}_{\mathrm{E}}(x,t)$, we define its linear impulse $\boldsymbol{I}$ to be given by the expression (10.2). It is then immediate on substituting in (10.4) from (9.5) that

$$\mathrm{d}\boldsymbol{I}/\mathrm{d}t = \kappa\oint \boldsymbol{V}_{\mathrm{E}} \wedge \boldsymbol{s}\,\mathrm{d}s. \tag{10.6}$$

Modifications to the definition of impulse are required if the vortex filament is infinitely long, but these are best treated in the context of the particular problem.

The angular impulse can be expressed as an integral over the vorticity distribution by the formula

$$\boldsymbol{A} = -\frac{1}{2}\int r^2\boldsymbol{\omega}\,\mathrm{d}V, \tag{10.7}$$

and can be considered in a similar manner. We find

$$\boldsymbol{A} = -\tfrac{1}{2}\kappa\oint \boldsymbol{R}^2\boldsymbol{s}\,\mathrm{d}s + \pi W a^2 \oint \boldsymbol{R}\wedge\boldsymbol{s}\,\mathrm{d}s + O(\rho). \tag{10.8}$$

It follows that

$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \kappa\oint \boldsymbol{R}\wedge\left(\frac{\partial\boldsymbol{R}}{\partial t}\wedge\boldsymbol{s}\right)\mathrm{d}s + 2\pi W a^2\oint\frac{\partial \boldsymbol{R}}{\partial t}\wedge\boldsymbol{s}\,\mathrm{d}s + \pi\frac{\mathrm{d}}{\mathrm{d}t}(Wa^2)\oint\boldsymbol{R}\wedge\boldsymbol{s}\,\mathrm{d}s. \tag{10.9}$$

We substitute for $\partial\boldsymbol{R}/\partial t$ from (9.5), and note that

$$\oint \boldsymbol{R}\wedge\frac{\partial}{\partial s}\left(\frac{\boldsymbol{b}}{\rho}\right)\mathrm{d}s = -\oint\frac{\boldsymbol{s}\wedge\boldsymbol{b}}{\rho}\,\mathrm{d}s = \oint\frac{\boldsymbol{n}}{\rho}\,\mathrm{d}s = 0,$$

$$\oint \boldsymbol{R}\wedge\frac{\boldsymbol{n}}{\rho}\,\mathrm{d}s = -\oint \boldsymbol{s}\wedge\boldsymbol{s}\,\mathrm{d}s = 0,$$

$$\oint \boldsymbol{R}\wedge\dot{\boldsymbol{s}}\,\mathrm{d}s = -\oint \boldsymbol{s}\wedge\left(\frac{\partial \boldsymbol{R}}{\partial t}+q\boldsymbol{s}\right)\mathrm{d}s-\frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}t}\oint \boldsymbol{R}\wedge\boldsymbol{s}\,\mathrm{d}s$$

$$= \oint \boldsymbol{V}_{\mathrm{E}}\wedge\boldsymbol{s}\,\mathrm{d}s+\frac{1}{2a^2W}\frac{\mathrm{d}}{\mathrm{d}t}(a^2W)\oint \boldsymbol{R}\wedge\boldsymbol{s}\,\mathrm{d}s$$

on using (8.17). Further

$$\oint \boldsymbol{R}\wedge\left[\mathrm{d}\boldsymbol{s}\wedge\int_{[\delta_0]}\mathrm{d}\boldsymbol{s}'\wedge\boldsymbol{\Delta}\right] = \iint[(\boldsymbol{R}\wedge\mathrm{d}\boldsymbol{s}')\,(\mathrm{d}\boldsymbol{s}\cdot\boldsymbol{\Delta})-(\boldsymbol{R}\wedge\boldsymbol{\Delta})\,\mathrm{d}\boldsymbol{s}\cdot\mathrm{d}\boldsymbol{s}'].$$

The second part of the integrand vanishes from symmetry. For the first, integrate by parts with respect to s which gives the integrand $\mathrm{d}\boldsymbol{s}'\wedge\mathrm{d}\boldsymbol{s}/|\boldsymbol{R}-\boldsymbol{R}'|$, that integrates to zero from symmetry. Hence, it follows after substitution that

$$\mathrm{d}\boldsymbol{A}/\mathrm{d}t = 0, \tag{10.10}$$

Again if $\boldsymbol{U}_{\mathrm{E}} \neq 0$, we define the angular impulse by (10.8) and it follows that

$$\mathrm{d}\boldsymbol{A}/\mathrm{d}t = \kappa\oint \boldsymbol{R}\wedge(\boldsymbol{V}_{\mathrm{E}}\wedge\boldsymbol{s})\,\mathrm{d}s. \tag{10.11}$$

The vanishing of $\mathrm{d}\boldsymbol{I}/\mathrm{d}t$ and $\mathrm{d}\boldsymbol{A}/\mathrm{d}t$ for an isolated filament provides a useful check on the analysis, and the conservation laws can provide a useful test of the accuracy of numerical investigation.

An expression for the kinetic energy as a line integral around the filament can also be constructed. We have

$$\text{K.E.} = \int \tfrac{1}{2}\boldsymbol{u}^2\,\mathrm{d}V = \int_{r>a}\tfrac{1}{2}\boldsymbol{u}^2\,\mathrm{d}V+\int_{r<a}\tfrac{1}{2}\boldsymbol{u}^2\,\mathrm{d}V, \tag{10.12}$$

where the first integral is over the region external to the core and the second one is over the core. Now, for an irrotational velocity field $\boldsymbol{u}$,

$$\int_{r>a}\tfrac{1}{2}\boldsymbol{u}^2\,\mathrm{d}V = \int_{r=a}\{\tfrac{1}{2}\boldsymbol{u}^2(\mathrm{d}\boldsymbol{S}\cdot\boldsymbol{x})-(\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{S})\,(\boldsymbol{u}\cdot\boldsymbol{x})\}, \tag{10.13}$$

if the fluid is unbounded and at rest at infinity. If there is more than one filament, the integral is over the surfaces of all the cores. If there are external walls, there is a similar integral over the external walls, the velocity in the integrand being given by the Biot–Savart law.

We evaluate (10.13), using the velocity field given by the sum of (5.8) and (5.15), and remembering that (in the notation of § 5)

$$\boldsymbol{x} = \boldsymbol{R}+\boldsymbol{r},$$

$$\mathrm{d}\boldsymbol{S} = -\boldsymbol{r}\left(1-\frac{\boldsymbol{r}\cdot\boldsymbol{n}}{\rho}\right)\mathrm{d}s\,\mathrm{d}\theta.$$

The result of a straightforward but tedious computation is

$$\int_{r>a}\tfrac{1}{2}\boldsymbol{u}^2\,\mathrm{d}V = \oint E_0\,\mathrm{d}s, \tag{10.14}$$

where
$$E_0 = -\kappa(\boldsymbol{R}\wedge(\boldsymbol{V}_{\mathrm{E}}+\boldsymbol{V}_{\mathrm{I}})\cdot\boldsymbol{s})-T_0\frac{\boldsymbol{n}\cdot\boldsymbol{R}}{\rho}+\frac{\kappa^2}{8\pi}\left(\frac{\boldsymbol{n}\cdot\boldsymbol{R}}{\rho}-2\right). \tag{10.15}$$

To the same order, the contribution from inside the core is

$$\int_{r<a} \tfrac{1}{2}\boldsymbol{u}^2\,\mathrm{d}V = \oint \tfrac{1}{2}\pi a^2(\overline{v^2}+\overline{w^2})\,\mathrm{d}s. \tag{10.16}$$

Since, $$\oint \frac{\boldsymbol{n}\cdot\boldsymbol{R}}{\rho}\left(\frac{\kappa^2}{8\pi}-T_0\right)\mathrm{d}s = \oint \boldsymbol{s}\cdot\frac{\partial}{\partial s}\left\{\left(T_0-\frac{\kappa^2}{8\pi}\right)\boldsymbol{R}\right\}\mathrm{d}s = \oint\left\{\left(T_0-\frac{\kappa^2}{8\pi}\right)+\boldsymbol{s}\cdot\boldsymbol{R}\frac{\kappa^2}{4\pi}\frac{\partial\rho}{\partial s}\right\}\mathrm{d}s,$$

we can write the total energy density per unit length as

$$E = T_0-\frac{3\kappa^2}{8\pi}-\kappa(\boldsymbol{R}\wedge(\boldsymbol{V}_{\mathrm{E}}+\boldsymbol{V}_{\mathrm{I}})\cdot\boldsymbol{s})+\frac{\kappa^2}{4\pi}\boldsymbol{s}\cdot\boldsymbol{R}\frac{\partial\rho}{\partial s}+\tfrac{1}{2}\pi a^2(\overline{v}^2+\overline{w}^2). \tag{10.17}$$

For a vortex ring of radius R, the kinetic energy is

$$2\pi RE = \tfrac{1}{2}\kappa^2 R\left(\ln\frac{8R}{a}-2\right)+\pi^2 Ra^2(\overline{v}^2+\overline{w}^2), \tag{10.18}$$

in agreement with the classical result (Saffman 1970).

Appendix A. The shear stress in a vortex filament

The Reynolds stress calculated in (6.5) is non-zero and parallel to $\boldsymbol{b}$ because the deformation of the streamlines caused by the curvature of the filament is symmetrical about $\boldsymbol{n}$ but not about $\boldsymbol{b}$. It is possible to calculate the deformation explicitly, this is straightforward but tedious, or to proceed indirectly as follows. We note that to $O(1/\rho)$, the core may be regarded as part of a circular vortex ring at rest in a uniform stream. We take cylindrical polar coordinates, x_1, along the axis of the ring, x_2 radial and x_3 azimuthal, with velocity components (u_1, u_2, w). To leading order, the equation of the ring is

$$x_1^2+(x_2-\rho)^2 = a_0^2. \tag{A 1}$$

Then $$\int w\boldsymbol{u}_\perp\,\mathrm{d}A = \int w(u_1,u_2)\,\mathrm{d}A,$$

and $\boldsymbol{b}$ is parallel to x_1 and $\boldsymbol{n}$ is parallel to $-x_2$. Now

$$\int wu_i\,\mathrm{d}A = \int\left[\frac{\partial}{\partial x_j}(wu_jx_i)-wx_i\frac{\partial u_j}{\partial x_j}-x_iu_j\frac{\partial w}{\partial x_j}\right]\mathrm{d}A, \tag{A 2}$$

where $j = 1, 2$. The equations of motion for steady motion of a ring give

$$\frac{\partial u_j}{\partial x_j} = -\frac{u_2}{x_2}, \quad u_j\frac{\partial w}{\partial x_j} = -\frac{u_2 w}{x_2}. \tag{A 3}$$

Hence $$\int wu_i\,\mathrm{d}A = -2\int\frac{u_2 w}{x_2}(x_1,x_2)\,\mathrm{d}A, \tag{A 4}$$

since the first term in (A 2) integrates to zero as the surface of the vortex is a streamline. In (A 4), we may substitute the leading order solution

$$w = w_0(r), \quad u_2 = (x_1/r)\,v_0(r), \tag{A 5}$$

for $$r^2 = x_1^2+(x_2-\rho)^2 \leqslant a_0^2,$$

giving $$\int wu_1\,\mathrm{d}A = \frac{2\pi}{\rho}\int_0^{a_0} r^2 v_0 w_0\,\mathrm{d}r, \quad \int wu_2\,\mathrm{d}A = 0, \tag{A 6}$$

the error being $O(1/\rho^2)$.

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Appendix B. Some kinematic relations

With $\boldsymbol{R} = \boldsymbol{R}(\xi, t)$, we have

$$\frac{\partial}{\partial s}\left(\frac{\partial \boldsymbol{R}}{\partial t}\right) = \frac{\partial \xi}{\partial s}\frac{\partial}{\partial \xi}\left(\frac{\partial \boldsymbol{R}}{\partial t}\right) = \frac{\partial \xi}{\partial s}\frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{R}}{\partial \xi}\right) = \frac{\partial \xi}{\partial s}\frac{\partial}{\partial t}\left(\boldsymbol{s}\frac{\partial s}{\partial \xi}\right) = \frac{\partial \boldsymbol{s}}{\partial t} + \boldsymbol{s}\frac{\partial}{\partial t}\left(\ln\frac{\partial s}{\partial \xi}\right). \tag{B 1}$$

Further, taking the scalar product of (B 1) with $\boldsymbol{s}$,

$$\frac{\partial}{\partial t}\left(\ln\frac{\partial s}{\partial \xi}\right) = \boldsymbol{s}\cdot\frac{\partial}{\partial s}\left(\frac{\partial \boldsymbol{R}}{\partial t}\right) = \frac{\partial}{\partial s}\left(\boldsymbol{s}\cdot\frac{\partial \boldsymbol{R}}{\partial t}\right) - \frac{\partial \boldsymbol{R}}{\partial t}\cdot\frac{\partial \boldsymbol{s}}{\partial s} = \frac{\partial V_{\parallel}}{\partial s} - \frac{\partial \boldsymbol{R}}{\partial t}\cdot\frac{\boldsymbol{n}}{\rho}, \tag{B 2}$$

where $V_{\parallel} = \boldsymbol{s}\cdot(\partial \boldsymbol{R}/\partial t) = \boldsymbol{s}\cdot\boldsymbol{V}$.

For any function $\phi(\xi, t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint \phi(\xi, t)\,\mathrm{d}s = \oint\left(\frac{\partial \phi}{\partial t} + \phi\frac{\partial}{\partial t}\ln\frac{\partial s}{\partial \xi}\right)\mathrm{d}s, \tag{B 3}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint \phi(\xi, t)\,\mathrm{d}\boldsymbol{s} = \oint\left(\frac{\partial \phi}{\partial t} + \phi\frac{\partial}{\partial t}\ln\frac{\partial s}{\partial \xi}\right)\mathrm{d}\boldsymbol{s} + \oint \phi\frac{\partial \boldsymbol{s}}{\partial t}\mathrm{d}s. \tag{B 4}$$

The velocity of a point on the vortex moving with constant s is

$$\left[\frac{\partial \boldsymbol{R}}{\partial t}\right]_s = \frac{\partial \boldsymbol{R}}{\partial t} - \boldsymbol{s}\frac{\partial s}{\partial t}. \tag{B 5}$$

Now

$$\frac{\partial s}{\partial t} = \int^{s}\left(\frac{\partial}{\partial t}\ln\frac{\partial s}{\partial \xi}\right)\mathrm{d}s = -q + \frac{s}{L}\frac{\mathrm{d}L}{\mathrm{d}t} \tag{B 6}$$

from (8.12), if we take $q = 0$ at $s = 0$. Hence

$$\left[\frac{\partial \boldsymbol{R}}{\partial t}\right]_s = \frac{\partial \boldsymbol{R}}{\partial t} + q\boldsymbol{s} - \frac{s}{L}\frac{\mathrm{d}L}{\mathrm{d}t}\boldsymbol{s}. \tag{B 7}$$

Then

$$\frac{\partial}{\partial s}\left[\frac{\partial \boldsymbol{R}}{\partial t}\right]_s = \frac{\partial}{\partial s}\left(\frac{\partial \boldsymbol{R}}{\partial t} + q\boldsymbol{s}\right) - \frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}t}\boldsymbol{s} - \frac{s}{L}\frac{\mathrm{d}L}{\mathrm{d}t}\frac{\boldsymbol{n}}{\rho}. \tag{B 8}$$

Appendix C. An integro-differential equation for long waves on a line vortex

The velocity of the centre of a vortex exhibiting helical oscillations (i.e. in the modes $m = \pm 1$) has components relative to fixed Cartesian axes

$$U(z,t) = \xi_+ \cos(\omega_+ t + \gamma z + \epsilon_+) + \xi_- \cos(\omega_- t + \gamma z - \epsilon_-), \tag{C 1}$$

$$V(z,t) = -\xi_+ \sin(\omega_+ t + \gamma z + \epsilon_+) + \xi_- \sin(\omega_- t + \gamma z - \epsilon_-). \tag{C 2}$$

Here, ξ_+ and ξ_- are constant amplitudes, ϵ_+ and ϵ_- are phase angles, and $\omega_+(\gamma)$ and $\omega_-(\gamma)$ is given by the dispersion relation (2.8) for $m = +1$ and $m = -1$, respectively. By virtue of equation (1.12),

$$\omega_-(\gamma) = -\omega_+(-\gamma), \tag{C 3}$$

and hence we can write (C 1) and (C 2) as

$$U(z,t) = \xi_+ \cos(\omega t + \gamma z + \epsilon_+) + \xi_- \cos(\omega t - \gamma z + \epsilon_-), \tag{C 4}$$

$$V(z,t) = -\xi_+ \sin(\omega t + \gamma z + \epsilon_+) - \xi_- \sin(\omega t - \gamma z + \epsilon_-), \tag{C 5}$$

where now ω stands for $\omega_+(\gamma)$ and is given by (2.8) or the generalization for non-uniform core structure given by the work in § 7, which in addition removes the restriction of displacement being small compared with core radius,

$$\frac{\omega}{\Omega}=\frac{1}{2}\left(a^2\gamma^2K-\frac{\overline{w^2}\gamma^2}{\Omega^2}\right)\left(1-\frac{\overline{w}\gamma}{\Omega}\right)-\lambda\frac{a^2\overline{w}\gamma^3}{\Omega}, \tag{C 6}$$

where here
$$K=\ln\frac{2}{|\gamma|\,a}-C+\frac{\overline{v^2}}{2\Omega^2a^2},\quad \Omega=\frac{\kappa}{2\pi a^2}. \tag{C 7}$$

The statements (C 4) and (C 5) are equivalent to the result that $U(z,t)$ and $V(z,t)$ satisfy the integro-differential equations

$$\frac{\partial U(z,t)}{\partial t}=\int_{-\infty}^{\infty}G(z-\zeta)\,V(\zeta,t)\,\mathrm{d}\zeta, \tag{C 8}$$

$$\frac{\partial V(z,t)}{\partial t}=-\int_{-\infty}^{\infty}G(z-\zeta)\,U(\zeta,t)\,\mathrm{d}\zeta, \tag{C 9}$$

where
$$G(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\omega(\gamma)\,\mathrm{e}^{\mathrm{i}\gamma u}\,\mathrm{d}\gamma. \tag{C 10}$$

It can be verified by direct substitution that (C 4) and (C 5) are the periodic solutions of (C 8) and (C 9).

The integrals are to be treated as generalized functions, since (C 10) does not converge in the usual sense. We have the results (Lighthill 1958)

$$\int\gamma^2\,\mathrm{e}^{\mathrm{i}\gamma\zeta}\,\mathrm{d}\gamma=-2\pi\delta''(\zeta),$$

$$\int_{-\infty}^{\infty}\gamma^2\ln|\gamma|\,\mathrm{e}^{\mathrm{i}\gamma\zeta}\,\mathrm{d}\gamma=\pi\frac{\mathrm{d}^3}{\mathrm{d}\zeta^3}\{(\ln|\zeta|+C)\,\mathrm{sgn}\,\zeta\},$$

and so on. If for simplicity we neglect the terms $O(\gamma^3)$ in (C 6), we find after some reduction that the integro-differential equations are

$$\frac{\partial U}{\partial t}=-\frac{\kappa}{8\pi}\int_{-\infty}^{\infty}\ln\frac{2|z-\zeta|}{a}\,\mathrm{sgn}\,(z-\zeta)\frac{\partial^3V}{\partial\zeta^3}\,\mathrm{d}\zeta-\frac{\kappa}{4\pi}\left(\frac{\overline{v^2}}{2\Omega^2a^2}-\frac{\overline{w^2}}{\Omega^2a^2}\right)\frac{\partial^2V}{\partial z^2}, \tag{C 11}$$

and
$$\frac{\partial V}{\partial t}=\frac{\kappa}{8\pi}\int_{-\infty}^{\infty}\ln\frac{2|z-\zeta|}{a}\,\mathrm{sgn}\,(z-\zeta)\frac{\partial^3U}{\partial\zeta^3}\,\mathrm{d}\zeta+\frac{\kappa}{4\pi}\left(\frac{\overline{v^2}}{2\Omega^2a^2}-\frac{\overline{w^2}}{\Omega^2a^2}\right)\frac{\partial^2U}{\partial z^2}. \tag{C 12}$$

The equations (C 8) and (C 9) can be derived from a variational principle. It can be verified that

$$\delta\iint\left\{(X_t^2+Y_t^2)+\frac{1}{2}\int G(z-\zeta)\,[X(z)\,Y_t(\zeta)-Y(z)\,X_t(\zeta)]\,\mathrm{d}\zeta\right\}\mathrm{d}z\,\mathrm{d}t=0 \tag{C 13}$$

gives (C 8) and (C 9) where
$$U=X_t,\quad V=Y_t. \tag{C 14}$$

Note that $X(z,t)$ and $Y(z,t)$ are the displacements of the centre of the vortex.

Appendix D. The effect of axial velocity on the stability of trailing vortices

Consider a pair of line vortices, of strengths $\pm\kappa$, whose parametric equations are

$$\left.\begin{aligned} \boldsymbol{R}_\mathrm{L} &= s\boldsymbol{i}-\tfrac{1}{2}h\boldsymbol{j},\\ \boldsymbol{R}_\mathrm{R} &= s\boldsymbol{i}+\tfrac{1}{2}h\boldsymbol{j}.\end{aligned}\right\} \tag{D 1}$$

The horizontal separation between the vortices is h, and a velocity $(\kappa/2\pi h)\,\boldsymbol{k}$ is superposed to keep the vortices at rest. We now employ our general equations of motion to study the stability of the vortex pair to infinitesimal disturbances. This problem was studied first by Crow (1970), using the cutoff approximation. Effects of axial velocity were incorporated by Widnall & Bliss (1971) and Parks (1971). There is general agreement on the shape of the stability boundary. This is confirmed by our analysis. However, significant disagreement on the wave speed of unstable disturbances exists, and we shall show that Widnall & Bliss are correct in order of magnitude, although their detailed prediction is in error.

The parametric equations of the perturbed vortices are taken as

$$\left.\begin{aligned} \boldsymbol{R}_\mathrm{L}(\xi,t) &= (\xi+x_\mathrm{L}(t)\,\mathrm{e}^{\mathrm{i}\gamma\xi})\,\boldsymbol{i}+(-\tfrac{1}{2}h+y_\mathrm{L}(t)\,\mathrm{e}^{\mathrm{i}\gamma\xi})\,\boldsymbol{j}+z_\mathrm{L}(t)\,\mathrm{e}^{\mathrm{i}\gamma\xi}\,\boldsymbol{k},\\ \boldsymbol{R}_\mathrm{R}(\xi,t) &= (\xi+x_\mathrm{R}(t)\,\mathrm{e}^{\mathrm{i}\gamma\xi})\,\boldsymbol{i}+(\tfrac{1}{2}h+y_\mathrm{R}(t)\,\mathrm{e}^{\mathrm{i}\gamma\xi})\,\boldsymbol{j}+z_\mathrm{R}(t)\,\mathrm{e}^{\mathrm{i}\gamma\xi}\,\boldsymbol{k}.\end{aligned}\right\} \tag{D 2}$$

The calculation now proceeds as follows. Focusing attention on the right-hand vortex, we first evaluate $\boldsymbol{V}_\mathrm{E}$ at a point ξ. This is the velocity induced by the perturbed left-hand vortex plus the uniform flow, and is calculated using the Biot–Savart law in the usual manner (see Crow (1970) for details). The self-induced velocity $\boldsymbol{V}_\mathrm{I}$ is now worked out from (4.7). Equation (9.9) is then used to evaluate $\dot{s}$. The change in length of the vortex is clearly of second order, and hence the changes in internal structure are also of second order and can be neglected. That is, the core radius a, the axial velocity W, the tension T, and the momentum flux parameter λ, may all be given their undisturbed values. Substitution into (9.5) now gives equations for the components of $\partial\boldsymbol{R}_\mathrm{L}/\partial t$. The details are rather involved and will not be given here. The final results are,† correct to the first order of small quantities,

$$\frac{\mathrm{d}x_\mathrm{R}}{\mathrm{d}t} = \frac{\mathrm{i}\kappa}{2\pi h^2}(\gamma h)^2K_1z_\mathrm{L}, \tag{D 3}$$

$$\frac{\mathrm{d}y_\mathrm{R}}{\mathrm{d}t} = \frac{\kappa}{2\pi h^2}[z_\mathrm{R}-\{\gamma hK_1+\gamma^2h^2K_0\}\,z_\mathrm{L}+\tfrac{1}{2}\gamma^2h^2\theta z_\mathrm{R}] + \mathrm{i}\gamma\frac{a^2}{h^2}W[y_\mathrm{R}-\gamma hK_1y_\mathrm{L}-\tfrac{1}{2}\gamma^2h^2\theta y_\mathrm{R}+\gamma^2h^2\lambda y_\mathrm{R}], \tag{D 4}$$

$$\frac{\mathrm{d}z_\mathrm{R}}{\mathrm{d}t} = \frac{\kappa}{2\pi h^2}[y_\mathrm{R}-\gamma hK_1y_\mathrm{L}-\tfrac{1}{2}\gamma^2h^2\theta y_\mathrm{R}] + \mathrm{i}\gamma\frac{a^2}{h^2}W[z_\mathrm{R}-\{\gamma hK_1+\gamma^2h^2K_0\}z_\mathrm{L}+\tfrac{1}{2}\gamma^2h^2\theta z_\mathrm{R}-\gamma^2h^2\lambda z_\mathrm{R}]. \tag{D 5}$$

Here K_0 and K_1 are modified Bessel functions of the second kind with argument γh. The quantity θ involves the internal structure and is given by

$$\theta = \ln\,(2/\gamma a_\mathrm{e})-C+\tfrac{1}{4}, \tag{D 6}$$

† The calculations were carried out by Mr H. Yuen of Caltech.

where $$a_e = a \exp\{\tfrac{1}{4} - (2\pi a^2/\kappa^2)(\overline{v^2} - 2\overline{w^2})\} \tag{D 7}$$

is the 'effective core radius'.

The complementary set of equations for the left-hand vortex are obtained by interchanging κ and $-\kappa$ and the subscripts L and R.

The transverse displacements $y_R, \ldots, z_L$, uncouple from the longitudinal displacement x_R, x_L. Following Crow, we can introduce the symmetric and antisymmetric modes,

$$\left.\begin{aligned} y_S &= y_R - y_L, \quad z_S = z_R + z_L, \\ y_A &= y_R + y_L, \quad z_A = z_R - z_L, \end{aligned}\right\} \tag{D 8}$$

which are independent.

Introducing disturbances proportional to $e^{\omega t}$, we find for the symmetric mode that

$$\omega^2 + i\gamma\omega \frac{a^2}{h^2} W(\phi - \psi - 2\gamma^2 h^2 \lambda) - \frac{\kappa^2}{4\pi^2 h^2}\phi\psi = 0, \tag{D 9}$$

where $$\left.\begin{aligned} \phi &= 1 - (\gamma h K_1 + \gamma^2 h^2 K_0) + \tfrac{1}{2}\theta\gamma^2 h^2, \\ \psi &= 1 + \gamma h K_1 - \tfrac{1}{2}\theta\gamma^2 h^2, \end{aligned}\right\} \tag{D 10}$$

and terms of relative order $\gamma^2 a^2$ have been neglected.

The stability boundary is given to the same order by

$$\phi\psi = 0, \tag{D 11}$$

which confirms the result given by Widnall & Bliss (1971) that the change in the stability boundary due to axial flow and internal structure is described by replacing a with a_e in Crow's (1970) results. The same is true for the wavelength of maximum amplification and the corresponding rate.

The travelling wave speed $c = -\operatorname{Im}\omega/\gamma$ of the unstable symmetric mode is

$$c = (a^2/h^2)\, W(\psi - \phi + 2\gamma^2 h^2 \lambda). \tag{D 12}$$

This contains an additional term $2\gamma^2 a^2 \lambda W$ when compared with the prediction of Widnall & Bliss. Note that γh is of order unity, so the additional term is comparable with the others. Parks (1971) predicted travelling wave speeds of order W, which is not in agreement with the present analysis.

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